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Generalized Symmetries in Condensed Matter

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Abstract

Recent advances in our understanding of symmetry in quantum many-body systems offer the possibility of a generalized Landau paradigm that encompasses all equilibrium phases of matter. This is a brief and elementary review of some of these developments.

1. EXTENDING THE LANDAU PARADIGM

If you have been to a condensed matter talk in the past few decades, you have seen the beating that Landau has been taking. The speaker begins by saying that Landau told us that states of matter are classified by the symmetries they break. After showing a picture of a donut, the speaker explains that in *this* talk, in contrast, they will discuss a state of matter that goes beyond Landau's limited conception of the world.

Having given such talks myself, I think it is extremely interesting that, in fact, with modern generalizations of our understanding of symmetry, it may be possible to incorporate all known equilibrium phases of matter into a suitably extended version of the Landau paradigm. Let me attempt to paraphrase the Landau paradigm:

1. Phases of matter should be labeled by how they represent their symmetries, in particular whether they are spontaneously broken or not.
2. The degrees of freedom at a critical point are the fluctuations of the order parameter.

A significant corollary of assertion 1 is that gapless degrees of freedom, or ground state degeneracy, in a phase, should be swept out by a symmetry. That is, they should arise as Goldstone modes for some spontaneously broken symmetry.

Beyond its conceptual utility, this perspective has a weaponization, in the form of Landau–Ginzburg theory, in terms of which we may find representative states, understand gross phase structure, and, when suitably augmented by the renormalization group (RG), even quantitatively describe phase transitions.

Indeed there are many apparent exceptions to the Landau paradigm. Let us focus first on apparent exceptions to assertion 1. As a preview, exceptions that are only apparent include the following:

- **Topologically ordered states.** These are phases of matter distinguished from the trivial phase by something other than a local order parameter (1, 2). Symptoms include a ground state degeneracy that depends on the topology of space, and anyons, excitations that cannot be created by any local operator. Real examples found so far include fractional quantum Hall states, as well as gapped spin liquids.
- **Other deconfined states of gauge theory.** This category includes gapless spin liquids such as spinon Fermi surface or Dirac spin liquids (most candidate spin liquid materials are gapless). Another very visible manifestation of such a state is the photon phase of quantum electrodynamics (QED) in which our vacuum lives.
- **Fracton phases.** Gapped fracton phases are a special case of topological order, where there are excitations that can be neither created nor moved by any local operator.
- **Topological insulators.** Here, we can include both free-fermion states with topologically nontrivial band structure and interacting symmetry-protected topological (SPT) phases.
- **Landau Fermi liquid.**

1.1. Conventions

L is the linear system size. $D = d + 1$ is the number of spacetime dimensions. I denote the dimension of a manifold or the degree of a form by a subscript or superscript. I also use fancy uppercase letters (like $\mathcal{A}_\mu$) for background gauge fields and lowercase letters (like a_μ) for dynamical gauge fields. I sometimes use $G^{(p)}$ to denote a p-form symmetry with group G.

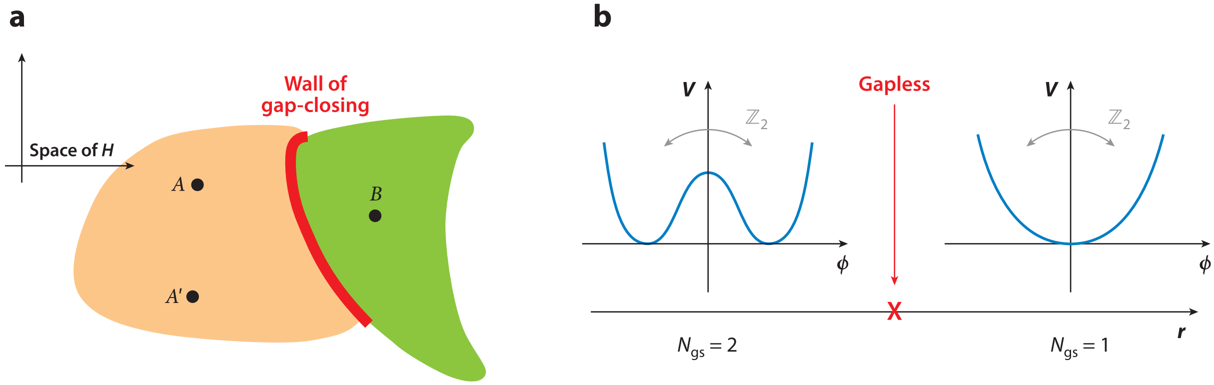


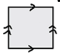
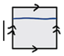
Figure 1

(*a*) A schematic illustration of the definition of gapped phases of matter. Two distinct phases are separated in the space of local Hamiltonians by a wall of gap-closing, the codimension-one locus where the gap closes. Here, $H_A \simeq H_{A'}$. (*b*) The ground-state degeneracy, N_{gs}, as swept out by a spontaneously broken global symmetry, is an example of a topological invariant that can label a phase.

1.2. Brief Nonsymmetry-Based Accounting of Gapped Phases

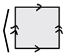
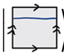
A useful definition of a gapped phase of matter is an equivalence class of gapped ground states of local Hamiltonians, in the thermodynamic limit.[1] Two ground states are considered equivalent if they are related by adiabatic evolution (for a time of order L^0) combined with inclusion or removal of product states. That is, there is a path between the two Hamiltonians along which the gap does not close (see **Figure 1*a***).

This definition poses a difficulty for checking that two Hamiltonians represent distinct phases: We cannot check all possible paths between them. A crucial role is therefore played by universal properties of a phase—quantities, such as integers, that cannot change smoothly within a phase and, therefore, can only vary across phase boundaries. A good example of such a topological invariant is the ground state degeneracy, which is certainly an integer. A phase of matter that spontaneously breaks a discrete symmetry G has a ground state degeneracy $|G|$, the order of the group (see **Figure 1*b***). This is a topological distinction from the trivial paramagnetic phase, which has a unique ground state and a representative that is a product state with no entanglement at all. In this sense, even spontaneous symmetry breaking (SSB) is a topological phenomenon.

Nontrivial phases can be divided into two classes: those with topological order and those without. One way to define topological order (1) is a phase with localized excitations that cannot be created by any local operator. In 2 + 1 dimensions, such particle excitations are called anyons; they can be created in pairs by an open-string operator. On a space with a noncontractible curve C, new ground states can be made by acting with the operator that transports an anyon around C. These ground states are locally indistinguishable, in the following sense. If $|$  $\rangle$ and $|$  $\rangle$

[1] We should pause to comment on the meaning of "gapped." We allow for a stable ground state subspace, which becomes degenerate in the thermodynamic limit. "Stable" means that the degeneracy persists under arbitrary small perturbations of the Hamiltonian and requires that the ground states are not related by the action of local operators. In d spatial dimensions, the logarithm of the number of such states can grow as quickly as L^{d-1} (3) in fracton models.

are two such ground states, then

$\Big\langle$  $\Big| \mathcal{O}_x \Big|$  $\Big\rangle = 0$ 1.

for all local operators $\mathcal{O}_x$. (The picture in the kets is a cartoon of two of the ground states on the two-torus.) A final symptom is the existence of long-range entanglement in the ground state; a review focusing on this aspect is Reference 4.

An interesting special case of topologically ordered states is fracton phases (5, 6). A fracton phase has excitations that cannot be moved by any local operator (perhaps only in some directions of space). This is a strictly stronger condition than topological order, because an excitation can effectively be moved by annihilating it and creating it again elsewhere. Such phases (with a gap) exist in $3 + 1$ dimensions (and higher). A consequence of the defining property is a ground state degeneracy whose logarithm grows linearly with system size and a subleading linear term in the scaling of the entanglement entropy of a region with the size of the region.

Even without topological order, there can be phases distinct from the trivial phase. One way in which they can be distinguished is by what happens if we put them on a space with boundary, so that there is a spatial interface with the trivial phase. A very rough (and not entirely correct) idea is that if the gap must close along the path to the trivial phase, then the coupling must pass through the wall of gap-closing at the edge of the sample. Phases that are distinguished in this way include integer quantum Hall states, topological insulators, and, more generally, SPT phases such as the Haldane phase of the spin-one chain or polyacetylene.

It seems that all of these examples transcend the Landau paradigm. My goal here is not to use the Landau paradigm as a straw man but rather to pursue it in earnest. The idea is that by suitably refining and generalizing our notions of symmetry, we can incorporate all of these "beyond-Landau" examples into a Generalized Landau Paradigm. There are two crucial ingredients, which work in concert: anomalies and generalized symmetries.

In this article, I speak of actual symmetries of physical systems, sometimes called "global symmetries." They act on the Hilbert space and take one state to another. In contrast, there is no such thing as gauge symmetry. In a gauge theory, the gauge invariance is a redundancy of a particular description of the system and is not preserved by relabeling degrees of freedom. For example, dualities (equivalences of physical observables at low energies) often relate a gauge theory with one gauge group to a gauge theory with a distinct gauge group. A familiar example in condensed matter physics is the duality between the XY model and the abelian Higgs model in $2 + 1$ dimensions (7, 8), but there are many others, e.g., Reference 9. This complaint about terminology hides an abyss of human ignorance: If someone hands you a piece of rock and asks whether its low-energy physics is described by some phase of a gauge theory, how will you tell? It is certainly true that phases realizable by gauge theory go beyond other constructions with only short-ranged entanglement; this begs for a characterization of these phases that transcends a description in terms of redundancies. Higher-form symmetries offer such a characterization for some such phases.

I want to highlight early attempts to understand topological order (10, 11), and the gaplessness of the photon (12) as consequences of generalized symmetry, as well as early appearances of generalized symmetries in the string theory literature (13–15). Other papers that have explicitly advocated for the utility of a Generalized Landau Paradigm include References 16–20.

2. HIGHER-FORM SYMMETRIES

The concept of higher-form symmetry that I review here was explained in References 16 and 21. It is easiest to introduce using a relativistic notation, so indices μ and ν run over space and time.

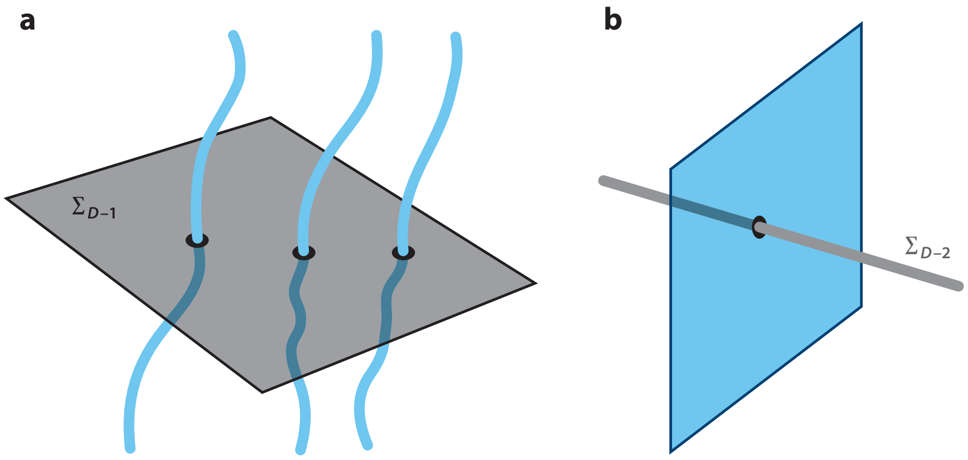


Figure 2

(*a*) In the case of an ordinary zero-form symmetry, the charge is integrated over a codimension-one slice of spacetime Σ_{D-1}, often a slice of constant time. All the particle worldlines (*blue curves*) must pass through this hypersurface. (*b*) The charge of a one-form symmetry is integrated over a codimension-two locus of spacetime Σ_{D-2} (a string in the case of $D = 2 + 1$). This surface intersects the worldsheets of strings (*blue sheet*).

Let's begin by considering the familiar case of a continuous zero-form symmetry. Noether's theorem guarantees a conserved current J_μ satisfying $\partial^\mu J_\mu = 0$. In the useful language of differential forms, this is $d\star J = 0$, where $\star$ is the Hodge duality operation.[2] This continuity equation has the consequence that the charge $Q_\Sigma = \int_{\Sigma_{D-1}} \star J$ is independent of the choice of time slice Σ. (Σ here is a closed d-dimensional surface, of codimension one in spacetime.) Notice that this is a topological condition. Q_Σ commutes with the Hamiltonian, the generator of time translations, and therefore so does the unitary operator $U_\alpha = e^{i\alpha Q}$, which we call the symmetry operator.[3]

If the charge is carried by particles, Q_Σ counts the number of particle worldlines piercing the surface Σ (as in **Figure 2*a***), and the conservation law $\dot{Q} = 0$ says that charged particle worldlines cannot end except on charged operators. If instead of a U(1) symmetry we only had a discrete $\mathbb{Z}_p$ symmetry, we could simply restrict $\alpha \in \{0, 2\pi/p, 4\pi/p \ldots (p-1)2\pi/p\}$ in the symmetry operator U_α. In that case, particles can disappear in groups of p.

Objects charged under a zero-form symmetry are created by local operators. Local operators transform under the symmetry by $\mathcal{O}(x) \to U_\alpha \mathcal{O}(x) U_\alpha^\dagger = e^{iq\alpha}\mathcal{O}(x)$, $d\alpha = 0$, where q is the charge of the operator. The infinitesimal version is $\delta\mathcal{O}(x) = \mathbf{i}[Q, \mathcal{O}(x)] = \mathbf{i}q\mathcal{O}(x)$.

Now let us consider a continuous one-form symmetry. This means that there is a conserved current that has two indices and is completely antisymmetric:

$$J_{\mu\nu} = -J_{\nu\mu} \text{ with } \partial^\mu J_{\mu\nu} = 0. \qquad 2.$$

We can regard J as a two-form and write the conservation law Equation 2 as $\mathrm{d}\star J = 0$, where d is the exterior derivative. As a consequence, for any closed codimension-two locus in spacetime Σ_{D-2}, the quantity $Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of Σ. The analog of the

[2]The Hodge dual of a p-form ω_p on a D-dimensional space with metric $g_{\mu\nu}$ has components $(\star\omega_p)_{\mu_1\cdots\mu_{D-p}} = \sqrt{\det g}\,\epsilon_{\mu_1\cdots\mu_D}\omega_p^{\mu_{D-p+1}\cdots\mu_D}$, where indices are raised with the inverse metric $g^{\mu\nu}$, and $\epsilon_{\mu_1\cdots\mu_D}$ is the antisymmetric Levi–Civita symbol.

[3]Throughout, I assume that the normalization is such that $Q \in \mathbb{Z}$, so that $\alpha \equiv \alpha + 2\pi$.

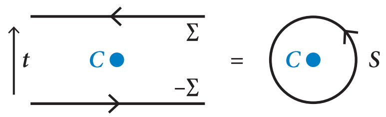


Figure 3

Equation 4 shows the expression for the transformation as $U(\Sigma)W(C)U\dagger(\Sigma)$. This operator ordering is obtained by placing the support of these operators on successive time slices. Because U is topological, from a spacetime point of view, the same result is obtained if instead we deform the surfaces Σ and $-\Sigma$ to a single surface S in spacetime that *surrounds* the locus C, as illustrated here in cross-section. The variation of the operator then depends on the linking number of S and C in spacetime.

symmetry operator is the unitary operator

$$U_\alpha(\Sigma) = e^{\mathbf{i}\alpha Q_\Sigma}. \tag{3}$$

Notice that reversing the orientation of Σ produces the adjoint of $U : U_\alpha(-\Sigma) = U_\alpha^\dagger(\Sigma)$.

The charge Q_Σ in the one-form case counts the number of charged string worldsheets intersecting the surface Σ (as in **Figure 2*b***). The conservation law Equation 2 then says that charged string worldsheets cannot end except on charged operators. The objects charged under a one-form symmetry are loop operators, $W(C)$. Fixing a constant-time slice M_{D-1}, such a loop operator transforms as

$$W(C) \to U_\alpha(\Sigma)W(C)U_\alpha^\dagger(\Sigma) = e^{\mathbf{i}\alpha \oint_C \Gamma_\Sigma} W(C), \quad \mathrm{d}\Gamma_\Sigma = 0. \tag{4}$$

Here, $\Sigma_{D-2} \subset M_{D-1}$ is any closed $(D-2)$-manifold, and Γ_Σ is its Poincaré dual in M_{D-1}, in the sense that $\int_{M_{D-1}} \eta^{(D-2)} \wedge \Gamma_\Sigma = \int_{\Sigma_{D-2}} \eta^{(D-2)}$ for all η; $\mathrm{d}\Gamma_\Sigma = 0$ because Σ has no boundary. The infinitesimal version of this transformation law is

$$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i}q\#(\Sigma, C)\, W(C), \tag{5}$$

where $\#(\Sigma, C)$ is the intersection number in M (see also **Figure 3**).

In the case of a discrete one-form symmetry, there is no current, but the symmetry operator $U_\alpha(\Sigma)$ is still topological. If the one-form symmetry group is $\mathbb{Z}_p$, strings can disappear or end only in groups of p.

For general integer $p \geq -1$, a p-form symmetry means the existence of topological operators $U_\alpha(\Sigma_{D-p-1})$ labeled by a group element α and a closed codimension-$(p+1)$ submanifold of spacetime.[4] For coincident submanifolds, these operators satisfy the "fusion rule" $U_\alpha(\Sigma)U_\beta(\Sigma) = U_{\alpha+\beta}(\Sigma)$. The operators charged under a p-form symmetry are supported on p-dimensional loci and create p-brane excitations. The conservation law asserts that the $(p+1)$-dimensional worldvolume of these excitations will not have boundaries.

For $p \geq 1$, the symmetry operators commute with each other—higher-form symmetries are abelian (16). To see this, consider a path integral representation of an expectation value with two symmetry operators $U(\Sigma_1)$ and $U(\Sigma_2)$ inserted on the same time slice t. The ordering of the operators can be specified in the path integral by shifting the left one to a slightly later time $t + \epsilon$. If $p \geq 1$, then $\Sigma_{1,2}$ have codimensions larger than one, and their locations can be continuously deformed to reverse their order.

2.1. Physics Examples of Higher-Form Symmetries

- Maxwell theory in $D = 3 + 1$ with electric charges but no magnetic charges has a continuous one-form symmetry with current $J^{\mu\nu}_{(m)} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \equiv \frac{1}{2\pi}(\mathrm{d}\tilde{A})^{\mu\nu}$. The statement that

[4]For discussion of $p = -1$, see Reference 22.

this current is conserved $\nabla_\mu J^{\mu\nu}_{(m)} = 0$ is the Bianchi identity expressing the absence of magnetic charge. The symmetry operator is $U^{(m)}_\alpha(\Sigma) = e^{\frac{\mathrm{i}\alpha}{2\pi}\int_\Sigma F}$. The fact that the charge operator $\int_\Sigma F$ depends only on the topological class of Σ is the magnetic Gauss law—when Σ is contractible, it counts the number of magnetic monopoles inside. This symmetry shifts the dual gauge field $\tilde{A}$ by a flat connection; the charged line operator is the 't Hooft line, $W^{(m)}[C] = e^{\mathrm{i}\oint_C \tilde{A}}$.

In free Maxwell theory without electric charges, there is a second one-form current, $J_{(e)} = F$, whose charged operator is the Wegner–Wilson line $W^{(e)}[C] = e^{\mathrm{i}\oint_C A}$. The symmetry operator for this "electric" one-form symmetry is $U^{(e)}_\alpha(\Sigma_2) = e^{\mathrm{i}\frac{2\alpha}{g^2}\int_{\Sigma_2}\star F}$, which (by canonical commutators) shifts the gauge field A by a flat connection.

- Pure $\mathsf{SU}(N)$ gauge theory or $\mathbb{Z}_N$ gauge theory or $\mathsf{U}(1)$ gauge theory with charge-N matter has a $\mathbb{Z}_N$ one-form symmetry, called the center symmetry. The charged line operator is the Wegner–Wilson line in the minimal irrep, $W[C] = \mathrm{tr} P e^{\mathrm{i}\oint_C A}$.
- When we spontaneously break a zero-form U(1) symmetry in $d = 2$, there is an emergent one-form U(1) symmetry whose charge counts the winding number of the phase variable φ around an arbitrary closed loop C, $Q[C] = \oint_C \partial\varphi$. In d spatial dimensions, this is a $(d-1)$-form symmetry. The charged operator creates a vortex (in $d = 2$, or a vortex line or sheet in $d > 2$). Unlike the examples above, this symmetry is generally not an exact symmetry of a microscopic Hamiltonian for a superfluid; it is explicitly broken by the presence of vortex configurations. This example and its consequences for superfluid physics are discussed further in Section 3.2.
- There is a sense in which the 3D Ising model has a $\mathbb{Z}_2$ one-form symmetry reflecting the integrity of domain walls between regions of up spins and regions of down spins. The charged line operator is the Kadanoff–Ceva disorder line (23)—the boundary of a region along which the sign of the Ising interaction is reversed (for a review, see Reference 24). But because a domain wall is always the boundary of some region, no states are charged; relatedly, the disorder line is not a local string operator. If we gauge the $\mathbb{Z}_2$ symmetry of the Ising model, the disorder line becomes the Wegner–Wilson line of the resulting $\mathbb{Z}_2$ gauge theory, and this theory has a genuine one-form symmetry.

2.2. Spontaneous Symmetry Breaking

Anything we can do with ordinary (zero-form) symmetries, we can do with higher-form symmetries. In particular, they can be spontaneously broken.

One way to characterize the unbroken phase of a zero-form symmetry is that correlations of charged operators are short-ranged, meaning that they decay exponentially with the separation between the operators

$$\left\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \right\rangle \sim e^{-m|x|}. \qquad 6.$$

In more general terms, we can regard the two points at which we insert a charged operator and its conjugate as an S^0, a zero-dimensional sphere, and the separation between the points as the size of the sphere. The broken phase for zero-form symmetry can be diagnosed by long-range correlations:

$$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger(x) \rangle \left\langle \mathcal{O}(0) \right\rangle + \cdots, \qquad 7.$$

independent of the size of the S^0.

For a p-form symmetry, the unbroken phase is also when correlations of charged operators are short-ranged, which decay when the charged object grows. For a one-form symmetry, this is when the charged loop operator exhibits an area law:

$$\langle W(C)\rangle \sim e^{-T_{p+1}\text{Area}(C)}, \tag{8.}$$

where Area(C) is the area of the minimal surface bounded by the curve C. In the case of electricity and magnetism, an area law for $\langle W^E(C)\rangle$ is the superconducting phase.

The broken phase for a p-form symmetry is signaled by a failure of the expectation value of the charged operator to decay with size. For a one-form symmetry, this is when the charged loop operator exhibits a perimeter law:

$$\langle W(C)\rangle = e^{-T_p\text{Perimeter}(C)} + \cdots . \tag{9.}$$

The coefficient T_p can be set to 0 by modifying the definition of $W(C)$ by counterterms local to C, so Equation 9 says that a large loop has an expectation value.

SSB of higher-form symmetry has been a fruitful idea. The fact that charged operators have long-range correlations means that the generators of the symmetry act nontrivially on the ground state. In the next two subsections, I illustrate the consequences in the cases of discrete and continuous symmetries, respectively. **Supplemental Material Section A** addresses the effects of fluctuations on higher-form order.

Supplemental Material >

2.3. Topological Order as Spontaneous Symmetry Breaking

One definition of topological order is the presence of a ground state subspace of locally indistinguishable states, as in Equation 1. This means that no local operator takes one ground state to another; instead the operator that takes one ground state to another is necessarily an extended operator. But this is equivalent to the spontaneous breaking of a higher-form symmetry (16, 25): The generators of the broken part of the higher-form symmetry commute with the Hamiltonian (at least at low energies) and act nontrivially on the ground states.

Let's think about the example of $\mathbb{Z}_p$ gauge theory (whose solvable limit is the toric code; 26) in D spacetime dimensions. This is a system with $\mathbb{Z}_p$ one-form symmetry with symmetry operators $U(M_{D-2})$, supported on a $(D-2)$-dimensional manifold, and charged operators $V(C_1)$, supported on a curve. In terms of the toric code variables, we can be completely explicit. On each link we have a p-state system on which act the Pauli operators $Z = \sum_{k=0}^{p} \omega^k \mid k\rangle\langle k|$ and $X = \sum_{k=0}^{p} \mid k+1\rangle\langle k|$ (where $\omega \equiv e^{2\pi \mathbf{i}/p}$ and the arguments of the kets are understood mod p). Then $V(C) = \Pi_{\ell\in C}X_\ell$ and $U(M) = \Pi_{\ell\perp M}Z_\ell$, where we regard M as a surface in the dual lattice, and $\ell\perp M$ indicates all links crossed by the surface M. The algebra of these operators is

$$U^m(M)V^n(C) = e^{2\pi\mathbf{i}\frac{mn}{p}\#(C,M)}V^n(C)U^m(M), \tag{10.}$$

where $\#(C,M)$ is the intersection number of the curve C with the surface M. This is the algebra of electric strings and magnetic flux surfaces in $\mathbb{Z}_p$ gauge theory. Deep in this gapped phase, $H = 0$, and there is a description in terms of topological field theory. A simple realization is BF theory of a one-form potential a and $(D-2)$-form potential b, with action

$$S[b,a] = \frac{p}{2\pi}\int_D b_{D-2}\wedge \mathrm{d}a, \tag{11.}$$

in terms of which

$$U^n(M) = e^{\mathbf{i}n\oint_M b_{D-2}},\quad V^m(C) = e^{\mathbf{i}m\oint_C a}. \tag{12.}$$

The algebra Equation 10 follows from canonical commutation relations in this Gaussian theory. Because $V(C)$ has a perimeter law in the deconfined phase, the charged objects whose condensation breaks the one-form symmetry are the lines of electric flux.

Another example is the Laughlin fractional quantum Hall states. So far the symmetry operators for a one-form symmetry with group A form a representation of A on the one-cycles of space, Z, i.e., a linear map $U : Z \to U(1)$, where the representation operators commute $U(M)U(M') = U(M')U(M)$. This relation can be generalized to allow for phases—i.e., a projective representation. Consider a system in $D = 2 + 1$ with a $\mathbb{Z}_k$ one-form symmetry that is realized projectively in the following sense:

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn \#(C,C')}{k}} U^n(C')U^m(C), \tag{13.}$$

where $\#(C, C')$ is the intersection number of the two curves C and C' in space. Regarding $U(C)$ as the holonomy of a charged particle along the loop C, this is the statement that flux carries charge. Representing this algebra nontrivially gives k ground states on T^2. This algebra, too, has a simple realization via abelian Chern–Simons (CS) theory, $S[a] = \frac{k}{4\pi} \int a \wedge \mathrm{d}a$, with $U^m(C) = e^{\mathbf{i}m \oint_C a}$.

The algebra in Equation 13 is a further generalization of one-form symmetry, in that the group law is only satisfied up to a phase. As I discuss in Section 3, it is an example of a one-form symmetry anomaly.

The preceding discussion applies to abelian topological orders. In this context, abelian means that the algebra of the line operators transporting the anyons forms a group, which must be abelian by the argument above. In Section 5, we discuss the further generalization that incorporates nonabelian topological orders.

2.4. Photon as Goldstone Boson

What protects the masslessness of the photon? The case of QED is the most visible version of this question; the same question arises in condensed matter as, Why are there $\mathsf{U}(1)$ spin liquid phases, with an emergent photon mode?

Higher-form symmetries provide a satisfying answer to this question (unlike appeals to gauge invariance, which is an artifact of a particular description): The gaplessness of the photon can be understood as being required by spontaneously broken $\mathsf{U}(1)$ one-form symmetry (12, 16, 17, 27), as a generalization of the Goldstone phenomenon.

Here is a perspective on the zero-form version of the Goldstone theorem. Given a continuous zero-form symmetry with current j_μ, we can couple it to a background gauge field $\mathcal{A}$ by adding to the Lagrangian $\Delta L \ni j_\mu \mathcal{A}^\mu$. If the symmetry is spontaneously broken, the effective Lagrangian will contain a Meissner term proportional to $\mathcal{A}^2$. But the effective action must be gauge invariant, and this requires the presence of a field that transforms nonlinearly under the $\mathsf{U}(1)$ symmetry: $\varphi \to \varphi + \lambda$ and $\mathcal{A} \to \mathcal{A} - \mathrm{d}\lambda$; this is a global symmetry if $\mathrm{d}\lambda = 0$. Altogether, the effective Lagrangian must contain a term of the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4\pi g} \left(\mathrm{d}\varphi + \mathcal{A}\right)^2, \tag{14.}$$

[where by $(\omega)^2$ I mean $\omega_p \wedge \star\omega_p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\omega^{\mu_1\cdots\mu_p}$]. The coefficient $\frac{1}{4\pi g}$ is the superfluid stiffness.

The analog for a continuous one-form symmetry works as follows. The current is now a two-form, so the background field must be a two-form gauge field $\mathcal{B}_{\mu\nu}$ and the coupling is $\Delta L \ni J_{\mu\nu}\mathcal{B}^{\mu\nu}$. The same logic implies that the effective action for the broken phase must contain a term,

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2g^2} \left(\mathrm{d}a + \mathcal{B}\right)^2, \tag{15.}$$

where the Goldstone mode a is a one-form that transforms nonlinearly, $a \to a + \lambda$ and $\mathcal{B} \to \mathcal{B} - \mathrm{d}\lambda$; this is a global symmetry if $\mathrm{d}\lambda = 0$. Setting the background field $\mathcal{B} = 0$, we recognize this as a Maxwell term for a. The coupling strength g is determined by the analog of the superfluid stiffness.

For p-form $\mathsf{U}(1)$ symmetry, we conclude by the same logic that there is a massless p-form field a with canonical kinetic term

$$S_{\mathrm{Max}}[a] = -\frac{1}{2g^2}\int \mathrm{d}a \wedge \star \mathrm{d}a. \qquad 16.$$

Returning to QED, we see that the familiar Coulomb phase is the SSB phase for the $\mathsf{U}(1)$ one-form symmetry. The unbroken phase is the superconducting phase of QED, where the photon has short-ranged correlations. (In an ordinary superconductor, where the Cooper pair has charge two, a $\mathbb{Z}_2$ subgroup of the one-form symmetry remains broken.)

As in the case of zero-form SSB, the broken phase can be understood via the condensation of charged objects; when the electric one-form symmetry is broken, the charged objects are the strings of electric flux (28, 29). Notice that the presence of charged matter, on which these strings can end, and which therefore explicitly breaks this symmetry, does not necessarily destroy the phase. We'll comment on this robustness more in Section 2.5. In fact, because of electromagnetic duality, the Coulomb phase is the broken phase for either the electric one-form symmetry or the magnetic one-form symmetry (16).

2.5. Robustness of Higher-Form Symmetries

We are used to the idea that consequences of emergent (aka accidental) symmetries are only approximate: explicitly breaking a spontaneously broken continuous zero-form symmetry gives a mass to the Goldstone boson.

This raises a natural question. The existence of magnetic monopoles with $m = M_{\mathrm{monopole}}$ explicitly breaks the one-form symmetry of electrodynamics: $\partial^\mu J^E_{\mu\nu} = j^{\mathrm{monopole}}_\nu$. If the photon is a Goldstone mode for this symmetry, does this mean the photon gets a mass? Perhaps surprisingly, the answer is no. This is a way in which zero-form and higher-form symmetries are quite distinct. The explanation of this statement is the subject of **Supplemental Material Section B**.

Supplemental Material >

2.6. Mean Field Theory

Landau–Ginzburg mean field theory is our zeroth-order tool for understanding symmetry-breaking phases and their neighbors. It is therefore natural to ask whether it has an analog for higher-form symmetries (30). We focus on the simplest case of a $\mathsf{U}(1)$ one-form symmetry.

It is worthwhile to review the logic that produces this weapon. If we take the Landau paradigm seriously, then the only low-energy modes we require are those swept out by the symmetry. The key idea is to introduce a degree of freedom $\phi(x)$ at each point in space that transforms linearly under the symmetry. ϕ should be regarded as a coarse-grained object, and this is an effective long-wavelength description. In the example of a magnet, $\phi(x)$ can be the magnetization averaged over a small cell at x. Now, because there are no other light degrees of freedom (by assertion 1), the effective action for ϕ should be given by an analytic functional of ϕ that is local in spacetime. This functional can therefore be expanded in a series consisting of all symmetric local functionals of ϕ, organized in a derivative expansion of terms of decreasing relevance. The length scale suppressing higher derivates is the short distance over which we averaged in constructing $\phi(x)$.

The one-form analog of the order parameter field $\phi(x)$ (which is a function from the space of points into a linear representation of G) is a functional, $\psi[C]$, from the space of loops into a linear representation of G, a "string field." Although $\phi(x)$ transforms under the zero-form symmetry as

$\phi(x) \to \phi(x)e^{i\alpha}$, with $d\alpha = 0$, the one-form analog transforms like $\psi[C] \to \psi[C]e^{i\oint_C \Gamma}$, with $d\Gamma = 0$.

Writing an action for such a field requires the analog of a derivative, which compares its values on nearby loops. Such an area derivative was discovered in the study of loop-space formulations of gauge theory (31; see **Supplemental Figure C.1**, *left*). The analog of integrating the action over spacetime $\int d^D x$ is integrating over the space of loops $\int [dC]$. The most general action consistent with the symmetries then takes the form

Supplemental Material >

$$S[\psi] = \int [dC] \left(V(|\psi[C]|^2) + \frac{1}{2L[C]} \oint ds \frac{\delta \psi^\star[C]}{\delta C_{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta C^{\mu\nu}(s)} + \cdots \right) + S_r[\psi]. \qquad 17.$$

The last recombination term

$$S_r[\psi] = \int [dC_{1,2,3}] \delta[C_1 - (C_2 + C_3)] \left(\lambda \psi[C_1] \psi^\star[C_2] \psi^\star[C_3] + h.c. \right) + \cdots \qquad 18.$$

is not local in loop space, but is local in real space because it involves only a single integral over the center of mass of the loops. Here, the delta function imposes the equality of loops regarded as integration domains (see **Supplemental Figure C.1**, *right*). The $\cdots$ denote terms with more derivatives or more powers of ψ. Models similar to this mean string field theory (MSFT) have been considered before in various specific contexts (32–36).

The potential term $V(|\psi[C]|^2) = r|\psi[C]|^2 + u|\psi[C]|^4 + \cdots$ controls the low-energy behavior. If $r > 0$, we find an unbroken phase, where $\psi[C] \simeq e^{-\sqrt{r}A[C]}$. When $r < 0$, the strings want to condense. The fluctuations around nonzero $|\psi|$ are all massive, except for the geometric mode $\psi[C] = v e^{i \oint_C ds a_\mu(x(s)) \dot{x}^\mu(s)}$, which describes a slowly varying one-form symmetry transformation and in terms of which the action of Equation 17 reduces to the Maxwell action for a, with coupling $g^2 = \frac{1}{2v^2}$.

As in the zero-form case, another application of this mean field theory is to classify topological defects of the resulting ordered media. The conclusion for $G = \mathsf{U}(1)$ is that the only topological defect is the codimension-three magnetic monopole. Further discussion of this MSFT, including comments about phase transitions, is deferred to **Supplemental Material Section C**.

3. ANOMALIES

My motivation for including a discussion of anomalies here is twofold. One is that anomalies are a necessary ingredient in a suitably Generalized Landau Paradigm that incorporates all phases, in particular topological insulators and SPT phases. A second motivation is that, as reviewed presently, the existence of anomalies makes symmetries much more useful for constraining the dynamics of a physical system, and their generalization to higher-form symmetries is therefore an essential step.

The historical, high-energy-physics perspective on anomalies starts from specifying a quantum field theory (QFT) by a path integral,

$$Z = \int [D(\text{fields})] e^{iS[\text{fields}]}. \qquad 19.$$

An anomaly is a symmetry of the action S that is not a symmetry of the path-integral measure. The first example found was the chiral anomaly, the violation of the axial current of a charged Dirac field (the symmetry that rotates left-handed and right-handed fermions with opposite phases),

$$\partial_\mu j^\mu_A = N \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \qquad 20.$$

which controls the decay of the neutral pion into two photons.

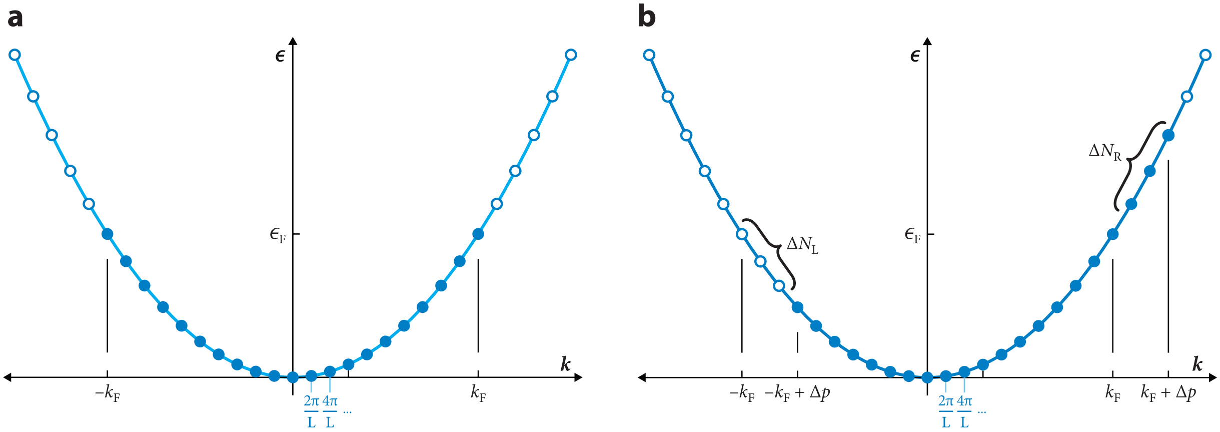


Figure 4

(*a*) Spectrum of a free-fermion tight-binding model in one dimension, near the bottom of the band at some small filling. Filled blue circles indicate filled states. (*b*) The result of adiabatically applying an electric field. N_L and N_R indicate the number of left-moving and right-moving excitations.

A more concrete perspective arises if we consider the same kind of system on the lattice, in one dimension for simplicity: Consider a tight-binding model of fermions hopping on a chain, at some small filling as in **Figure 4**. In this case, there is no chiral symmetry at all at the lattice scale. It is an emergent symmetry, violated by the UV physics in a definite way. At low energies, the system is approximately described by the neighborhood of the two boundaries of the Fermi sea, giving a 1D massless Dirac fermion with a chiral symmetry. But if we adiabatically apply an electric field E_x, every fermion increases its momentum and the chiral charge changes by

$$\Delta Q_A = \Delta(N_R - N_L) = 2\frac{\Delta p}{2\pi/L} = \frac{L}{\pi}e\int \mathrm{d}t E_x(t) = \frac{e}{2\pi}\int \epsilon_{\mu\nu}F^{\mu\nu}. \qquad 21.$$

The left hand side is $\Delta Q_A = \int \partial^\mu j_\mu^A$, and so Equation 21 is the 2D version of the chiral anomaly:

$$\partial_\mu j_A^\mu = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}. \qquad 22.$$

A reason for excitement about this phenomenon is that the coefficient N in Equation 20 is an integer. This is the first hint that an anomaly is a topological phenomenon, a quantity that is RG invariant (37). The idea is that the existence of the anomaly means that the partition function varies by some particular phase under the anomalous symmetry, but an RG transformation must preserve the partition function. Much of physics is about trying to match microscopic (UV) and long-wavelength (IR) descriptions. That is, we are often faced with questions of the form, “What could be a microscopic Hamiltonian that produces these phenomena?,” and “What does this microscopic Hamiltonian do at long wavelengths?” Anomalies are precious to us, because they are RG-invariant information: Any anomaly in the UV description must be realized somehow in the IR description.

Another useful perspective on anomaly is as an obstruction to gauging the symmetry. Gauging a symmetry means creating a new system in which the symmetry is a redundancy of the description by coupling to gauge fields. If the symmetry is not conserved in the presence of background gauge fields, the resulting theory would be inconsistent.

An example of an anomaly of a continuous symmetry is described above. Discrete symmetries can also be anomalous.

Anomaly is actually a more basic notion than phase of matter: The anomaly is a property of the degrees of freedom (of the Hilbert space) and how the symmetry acts on them, independent of a choice of Hamiltonian. Multiple phases of matter can carry the same anomaly.

3.1. Symmetry-Protected Topological Phases and Anomalies

The definition of gapped phases can be refined by studying only the space of Hamiltonians preserving some particular symmetry group G. Two phases that are distinct in this smaller space may nevertheless be connected by a gapped path in the larger space of nonsymmetric Hamiltonians.

One way to define (38) an SPT phase is as a nontrivial phase of matter (with some symmetry G) without topological order (for a review, see Reference 39). SPT phases can be characterized by their edge states. The idea is that the edge theory has to represent an anomaly of the symmetry G. It is really this anomaly that labels the bulk phase. This phenomenon is called anomaly inflow.

As a simple example, consider an effective field theory for the integer quantum Hall effect, regarded as an SPT for charge conservation symmetry.[5] The charge conservation symmetry is associated, by Noether's theorem, with a conserved current j^μ, with $\partial_\mu j^\mu = 0$. In $D = 2 + 1$, this equation can be solved by writing $j^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho/(2\pi)$, in terms of a one-form gauge field a_μ, with redundancy $a \to a + \mathrm{d}\alpha$. The leading effective action for such a field, in the absence of parity symmetry, is a CS term (40, 41):

$$S_{\mathrm{IQH}}[a, \mathcal{A}] = \frac{1}{4\pi}\int_M \epsilon^{\mu\nu\rho}\left(a_\mu\partial_\nu a_\rho + 2\mathcal{A}_\mu\partial_\nu a_\rho\right), \qquad 23.$$

where $\mathcal{A}$ is a background field for the charge conservation symmetry. Under $\mathcal{A} \to \mathcal{A} + \mathrm{d}\lambda$, $\delta S_{\mathrm{IQH}} = \int_{\partial M}\frac{\epsilon^{ij}}{4\pi} f_{ij}\lambda$. This is the contribution to the chiral anomaly from a single right-moving edge mode.

In terms of the definition of the anomaly as a variation of the partition function of the edge theory in the presence of background fields, the variation of the bulk action cancels the anomaly of the edge theory, so that the whole system is G symmetric. The edge theory cannot be trivial, because it has to cancel the variation of the bulk under the symmetry transformation: It has to be one of the following (42):

- gapless,
- symmetry-broken, or
- topologically ordered.

In particular, there cannot be a trivial gapped ground state. These are the same conditions arising from the Lieb–Schultz–Mattis–Oshikawa–Hastings (LSMOH) theorem (43, 44; for more recent developments, see, e.g., Reference 45), and we can call this an LSMOH constraint.

A perhaps simpler example is the free fermion topological insulator in $D = 3 + 1$, protected by charge conservation and time-reversal symmetry. In this case, the bulk effective action governs a single massive Dirac fermion; a boundary is an interface where the mass changes sign, at which a single Dirac cone arises. A single Dirac cone in $D = 2 + 1$ realizes the so-called parity anomaly. The fact that anomaly transcends a phase of matter is illustrated by the fact that, in the presence of interactions or disorder, there are other possible edge theories for the topological insulator.

[5]Actually, the integer quantum Hall phase is more robust and survives explicit breaking of the charge conservation symmetry. It is protected by the gravitational anomaly manifested in the nonzero chiral central charge.

There is by now a sophisticated (still conjectural) mathematical classification of SPTs for various G in various dimensions (46, 47) about which I will not say more here. My point is that we are still labeling these phases by their realization of symmetries!

3.2. Anomalies of Higher-Form Symmetries

Let's return to the example from Section 2.1 of the $(d-1)$-form symmetry that arises in any superfluid phase (17, 18, 48). The current can be written as $(\star J)_\mu = \partial_\mu \varphi$. However, in the presence of a background gauge field A for the $\mathsf{U}(1)$ symmetry, the gauge-invariant current is instead

$$(\star J)_\mu = D_\mu \varphi, \tag{24.}$$

where $D_\mu \varphi = \partial_\mu \varphi - qA_\mu$ is the covariant derivative. But this current is not conserved:

$$\mathrm{d} \star J = -qF, \tag{25.}$$

with $F \equiv \mathrm{d}A$. This equation has a simple interpretation: Applying an electric field leads to a supercurrent that increases linearly in time.

The symmetry violation in Equation 25 is an example of a mixed anomaly between a zero-form symmetry and a $(d-1)$-form symmetry that arises automatically from SSB. Reference 18 shows a converse statement: Any system with $\mathsf{U}(1)^{(0)} \times \mathsf{U}(1)^{(D-2)}$ symmetry with an anomaly of the form in Equation 25 contains a Goldstone boson in its spectrum. Because no long-range order is assumed, this is a more general statement than Goldstone's theorem—it applies even in $D=2$. This perspective can be used to demonstrate the existence of equilibrium states with nondissipating current (48).

A direct one-form generalization of Oshikawa's argument (43) appears in Reference 49. This is an example of a mixed anomaly between a one-form symmetry and lattice translation symmetry.

We should give an example of an anomaly of a higher-form symmetry that does not involve zero-form symmetries. An example is provided by the theory of abelian anyons in $D = 2 + 1$ and is best understood by regarding an anomaly as an obstruction to gauging. Gauging a continuous one-form symmetry means coupling the conserved current $J^{\mu\nu}$ to a dynamical two-form gauge field, $b_{\mu\nu}$, by a term like $b_{\mu\nu}J^{\mu\nu}$. That is, gauging a symmetry means summing over all possible background fields. In the discrete case, this is the same as summing over the insertions of all possible symmetry operators. (In the continuous case, it also requires summing over connections that are not flat.)

Thus, gauging a one-form symmetry in $2 + 1$ dimensions means proliferating the worldlines of the associated anyons (16, 50); this is anyon condensation (51). But it only makes sense to condense particles with bosonic self-statistics: Condensation means essentially that the many-particle wave function is a constant, which has bosonic statistics. Therefore, a subgroup of a one-form symmetry generated by line operators with nontrivial statistics cannot be gauged. We conclude that, in $2 + 1$ dimensions, the 't Hooft anomaly of a one-form symmetry is encoded in the self-statistics of the line operators, i.e., of the anyons. Thus, the algebra Equation 13 is an example of a one-form symmetry with an 't Hooft anomaly. Notice that from this point of view, nontrivial mutual statistics of a pair of anyon types a and b is a mixed 't Hooft anomaly: It does not stop us from gauging (i.e., condensing) a, but we cannot condense both simultaneously, because in the presence of the a condensate, b is confined. The algebra for discrete gauge theory (Equation 10) can also be regarded an example of an anomaly for higher-form symmetry because the charged operators V_n are also topological; so this is a one-form symmetry and a $(D-2)$-form symmetry with a mixed anomaly.

3.3. Symmetry-Protected Topological Phases of Higher-Form Symmetries

We can combine the ingredients of the above discussions and consider SPT phases that are protected by higher-form symmetries (21). This is a slightly awkward subject because higher-form symmetries tend to be emergent, and it therefore might be artificial to restrict ourselves to the subspace of Hamiltonians with exact higher-form symmetry.

In $D = 2 + 1$, an 't Hooft anomaly for a one-form symmetry is diagnosed by the self-statistics of the line operators. So the edge of a one-form G SPT in $D = 3 + 1$ just needs to have G topological order with quasiparticles that aren't bosons. Lattice models for higher-form SPTs have been written down in References 52 and 53, and effective theories were studied in Reference 54.

4. SUBSYSTEM SYMMETRIES AND FRACTON PHASES

Above we have discussed p-form symmetries, described by symmetry operators acting on codimension-$(p + 1)$ submanifolds of spacetime. These operators were deformable, in the sense that their correlations only depend on their deformation class in spacetime (avoiding any charged operator insertions).

A distinct generalization of the notion of symmetry arises by defining symmetry operators acting independently on rigid subspaces of the space on which the system is defined. That is, we can imagine that there is a different symmetry operator for each subspace, even in the same homology class, so that the symmetry operators are not topological, but still commute with the Hamiltonian. This is sometimes called a "faithful" symmetry (55) or subsystem symmetry. This generalization is not compatible with Lorentz invariance.

An object charged under such a subsystem symmetry cannot leave the locus on which the symmetry is defined. This sort of restricted mobility of excitations is a defining property of fracton phases (5, 6). A fracton phase can be identified as one that spontaneously breaks such a faithful higher-form symmetry (55–57). Foliated fracton phases (58) like the X-cube model (59) spontaneously break a "foliated one-form symmetry" acting independently on each plane of a lattice (55).

A closely related concept is that of multipole symmetries (e.g., 60–65). A multipole symmetry is one in which the continuity equation involves extra derivatives, like $\partial_0 J^0 + \partial_i \partial_J J_{ij} = 0$ (a dipole symmetry). Such a conservation law produces conserved charges that need not be integrated over all of space and act independently of each other. [For example (62), consider the continuity equation $\partial_0 J^0 + \partial_x \partial_y J = 0$ in $D = 2 + 1$; then $Q_x(x) = \int dy J^0(x, y)$ is conserved for each x.] The simplest example is that conservation of dipole moment implies that charges are immobile (60).

Models with such symmetries have been studied for a long time in the condensed matter literature (66). Efforts to understand how the rules of ordinary field theory must be relaxed to accommodate such systems and their symmetries have been vigorous (see, e.g., 58, 62–64, 67–71, and references therein and thereto). Attempts have been made to classify subsystem-symmetry-protected topological phases (72) and their anomalies (73), and to understand subsystem-symmetry-enriched topological order (74). A subsystem-symmetry-based understanding of Haah's code (75) appears in Reference 76.

An important issue is the robustness of such phases, especially in the gapless case, upon breaking the large symmetry group. At least in examples, the scaling dimensions of operators charged under the subsystem symmetry is large, and in fact diverges in the continuum limit (62–64, 66, 70; see, in particular, Reference 66, their equation 121). This shows that there is at least a small open set in the space of subsystem-symmetry-breaking couplings in which such phases persist.

The subsystem on which a symmetry acts can be more interesting than just a line or a plane. For example, it can be a fractal (77, 78). The Newman–Moore model (79) is a simple example of

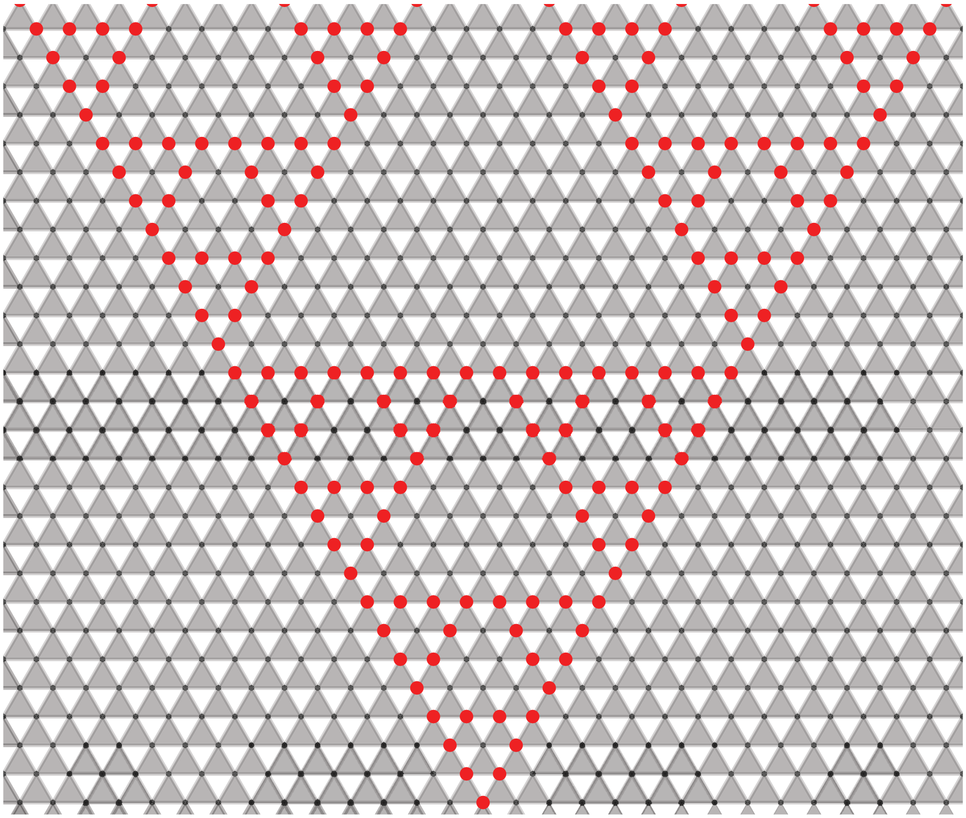


Figure 5

An example of the support of a fractal symmetry operator in the Newman–Moore model. If we flip only the red spins, it preserves the Hamiltonian Equation 26. That is, every up-triangle has an even number of red dots. There are many ways to accomplish this.

a model with a symmetry operator supported on a fractal subset of space. Put qubits on the sites i of the triangular lattice and consider,

$$H = \sum_{ijk \in \Delta} Z_i Z_j Z_k + g \sum_i X_i, \tag{26.}$$

where the sum is only over up-pointing triangles. To see that this has a fractal symmetry, pick a spin to flip, say, the circled spin in **Figure 5**. Moving outward from that starting point and demanding that each up-triangle contains an even number of flipped spins, there are many possible self-similar subsets of the lattice we can choose to flip. In fact, there is an extensive number.

This transverse-field Newman–Moore model (Equation 26) has a number of interesting properties. It has a self-duality mapping $g \to 1/g$, obtained by defining dual spins $\tilde{X}_\Delta \equiv \prod_{i \in \Delta} Z_i Z_j Z_k$ on a new lattice with sites corresponding to the up-pointing triangles. The exotic critical point at $g = 1$[6] (81) separates a gapped paramagnetic phase from a gapless phase in which the fractal $\mathbb{Z}_2$ symmetry is spontaneously broken. Other models with such fractal symmetry have been studied in Reference 82.

[6] Earlier work (77, 80) found indications of a first-order transition.

5. CATEGORICAL SYMMETRIES

Our understanding of what is a symmetry of a quantum many-body system or QFT has evolved quite a bit. The above discussion shows that the presence of a symmetry means the existence of topological defect operators.[7] (I believe the word 'defect' in this name just refers to the fact that these operators have positive codimension.) In the case of an ordinary symmetry, these are the symmetry operators, $U_g(\Sigma_d)$, that we discussed above; they are labeled by a group element $g \in G$, and supported on a codimension-one (e.g., fixed-time) slice Σ_d, and they are topological in the sense that their correlation functions do not change under continuous deformations. These operators satisfy a fusion rule in the sense that for two symmetry operators associated with the same time slice, $\lim_{\epsilon\to 0^+} U_g(t+\epsilon)U_h(t) = U_{gh}(t)$. When a local operator crosses such a U_g, it gets acted on by the transformation g.

If the surface Σ_d is not a fixed-time slice, such an operator implements a modification of the Hamiltonian, such as a change of boundary conditions. A good example to keep in mind is the defect operator $U_{-1}(\Sigma)$ in the classical Ising model. It is an instruction to flip the sign of the coupling along any bond crossing the codimension-one locus Σ. This operator is topological: Deforming Σ through a region R is accomplished by redefining all the spins in R by $\sigma \to -\sigma$. This shows that the charged operator is the spin.

A useful perspective is to reverse the logic and regard the existence of topological defect operators as the definition of a symmetry. One of the advantages of this perspective is that it treats continuous and discrete symmetries uniformly; it also makes no reference to transformations of fields, and so treats Noether symmetries and topological symmetries uniformly. And from this perspective it is easy to see some generalizations. The first generalization is that a p-form symmetry is associated with (unitary) topological operators whose support has codimension $p+1$. In the low-energy theory describing an abelian topological order in $D = 2+1$, these operators U are the holonomies of anyon worldlines. For two operators on the same submanifold, $U_\alpha(M_{D-p-1})U_\beta(M_{D-p-1}) = U_{\alpha+\beta}(M_{D-p-1})$, where for $p > 1$ the order does not matter, and we adopt an additive notation.

The preceding discussion suggests a further generalization, which we need in order to describe nonabelian topological order as SSB: What about the worldlines of nonabelian anyons? This is a dramatic step because the algebra of topological operators T_a that transport nonabelian anyons is no longer a group. Rather, they satisfy the fusion algebra:

$$T_a T_b = \sum_c N_{ab}^c T_c. \qquad 27.$$

By definition, a topological order is nonabelian if there is more than one term on the right-hand side of this equation for some choice of a and b. Whereas multiplication of two elements of a group always produces a unique third element, here we produce a superposition of elements, weighted by fusion multiplicities N_{ab}^c. Furthermore, there is some tension between the fusion algebra (Equation 27) and unitarity of the operators T_c. The trivial anyon corresponds to the identity operator, $T_1 = 1$. Each type of anyon a has an antiparticle $\bar{a}$. Because $T_{\bar{a}}$ corresponds to transporting a in the opposite direction, we expect that $T_{\bar{a}} = T_a^\dagger$, and therefore Equation 27 says, in particular,

$$T_a T_a^\dagger = \sum_c N_{a\bar{a}}^c T_c. \qquad 28.$$

[7]A sufficient condition for this conclusion is Lorentz symmetry. In its absence, we have already seen examples of systems with subsystem symmetries, where there are operators that commute with the Hamiltonian that are not fully topological.

If the right-hand side here has a term other than $N^1_{a\bar{a}}$, then T_a is not unitary. As an example, consider the Ising topological order, with three anyon types $\{1, \psi, \sigma\}$ and the fusion rules

$$T_\sigma T_\sigma = 1 + T_\psi, \quad T_\sigma T_\psi = T_\psi T_\sigma = T_\sigma, \quad T_\psi T_\psi = T_1. \qquad 29.$$

Note that σ is its own antiparticle. Equation 29 implies that the topological line operator T_σ cannot be unitary and, furthermore, cannot be inverted by any linear combination of T_a. Such symmetries are called categorical symmetries or fusion category symmetries.

More generally, any algebra of topological operators acting on a physical system can be regarded as encoding some kind of generalized symmetry. At the moment, condensed matter applications of the idea of fusion category symmetries remain in the realm of relatively formal developments, as opposed to active phenomenology of real materials. One application is to understand nonabelian topological order as SSB.[8] A concrete example of a model with noninvertible symmetries is G_k CS theory, with nonabelian gauge group G at level $k > 1$. The noninvertible symmetry operators are the Wegner–Wilson lines. The specific example of $\mathsf{SU}(2)_2$ CS theory can describe the Ising topological order and is possibly realized as part of the effective low-energy description of $\nu = \frac{5}{2}$ quantum Hall states.

More generally, any topological field theory for nonabelian topological order enjoys such a noninvertible symmetry. A nice example of the application of this perspective on anyon worldlines as symmetry operators is in Reference 85, which provides a condition on the anyon data required for a general (2 + 1)D topological order to admit a gapped boundary condition, beyond vanishing chiral central charge.

Part of the reason for the name "categorical symmetry" is that such a collection of symmetry operators comes with some additional data. Besides putting two symmetry operators right on top of each other, we can also consider symmetry operators associated with branched manifolds, as in **Figure 6*a***. Once we allow such objects, we must also consider more complicated objects related to the associativity of the product, as in **Figure 6*b***, which relates the two ways of resolving a four-valent junction of topological operators into two three-valent junctions. This associativity information (creatively called F-symbols) is part of the specification of the categorical symmetry and must satisfy the pentagon identities (see, e.g., Reference 86, their figure 1). In the case of one-form symmetry in (2 + 1) dimensions there is further information associated with braiding.

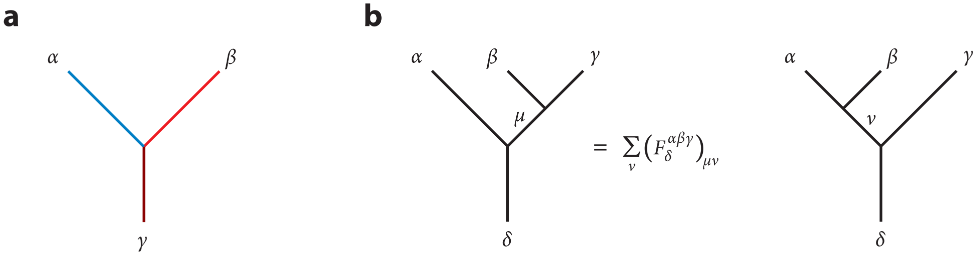


Figure 6

(a) Fusion of symmetry operators: This junction is allowed if $N^\gamma_{\alpha\beta} \neq 0$. ($b$) Associativity data of fusion of symmetry operators (in the simpler case where the fusion coefficients $N^\gamma_{\alpha\beta}$ are only 0 or 1).

[8]A related perspective appears via the "pulling-though" operators in the tensor network description of topological orders reviewed in Reference 83. For a study of categorical symmetries realized as matrix product operators, see Reference 84.

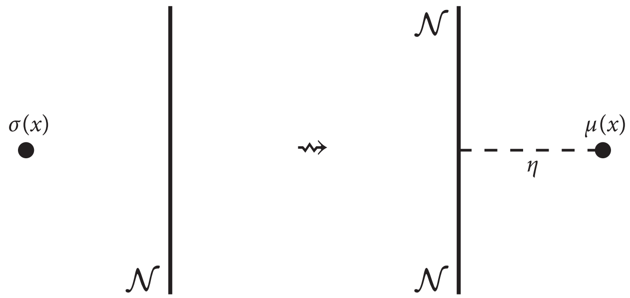


Figure 7

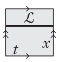
When a spin σ moves through the duality wall $\mathcal{N}$, it turns into a disorder operator μ, attached by a topological line η to the duality wall. The right figure illustrates the fact that $N^{\eta}_{\mathcal{N}\mathcal{N}} \neq 0$.

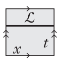
A good example of a noninvertible line operator appears in the critical Ising conformal field theory (CFT) in $D = 1 + 1$, in the form of the duality wall (**Figure 7**). The definition of such an object entails the following: When we pass through the wall, we act by the Kramers–Wannier self-duality interchanging the spin and the disorder operator. The latter is not a local operator but rather must be attached to a branch cut, across which the $\mathbb{Z}_2$ symmetry acts. Moving a local spin operator through such a duality wall then turns it into an operator attached by a topological defect line to the duality wall. The fusion algebra of the duality wall operator $\mathcal{N}$ and the ordinary $\mathbb{Z}_2$ symmetry line operator η can be summarized as

$$\eta\eta = 1,\ \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N},\ \mathcal{N}\mathcal{N} = 1 + \eta.$$

(These are a relabeling of the Ising fusion rules above.) The last, nonabelian, relation comes from the fact that the Kramers–Wannier duality only keeps track of the locations of domain walls and erases the information about the overall spin flip. In a theory with such a symmetry operator, RG flows generated by a perturbation by a local operator can only generate operators that pass freely through the wall (87, 88). Examples of duality walls in $D = 3 + 1$ were studied in References 89 and 90.

Categorical symmetries have been studied most extensively in (1 + 1)D QFTs (e.g., 86–88, 91–94), where they can be used to constrain RG flows. It was shown in Reference 87 that certain noninvertible symmetries can forbid a trivial gapped ground state, as in the LSMOH theorem. The idea is to consider the partition function on T^2 with a symmetry line operator $\mathcal{L}$ wrapping one of the circles, and argue by contradiction. If there is a gap, we can evaluate this quantity in the effective low-energy topological theory. Demanding modular invariance (i.e., that we get the same answer whichever circle we regard as time) relates the trace over the Hilbert space with twisted boundary conditions,

$$\mathrm{tr}_{\mathcal{H}_{\mathcal{L}}} e^{-\beta(H-E_0)} =$$ 

$=$ 

$$= \mathrm{tr}\mathcal{L}e^{-\frac{2\pi}{\beta}(H-E_0)}, \qquad 30.$$

to the ordinary trace with the insertion of the symmetry operator. In a topological field theory, the former quantity is just $\mathrm{tr}_{\mathcal{H}_{\mathcal{L}}}1$, the number of states in the twisted sector. If there were furthermore a unique ground state, then the latter quantity would be just $\langle\mathcal{L}\rangle$. Because the former is a nonnegative integer, we can conclude that if $\langle\mathcal{L}\rangle$ is not a nonnegative integer, then there cannot be a unique gapped ground state. For example, a certain perturbation of the tricritical Ising model has a symmetry operator W with (Fibonacci) fusion algebra $W^2 = 1 + W$. This algebra implies that the eigenvalues of W are $(1 \pm \sqrt{5})/2$, and there must therefore be an even number of vacua for its expectation value to be an integer. A related argument shows (95) that all irreps of

G appear in the spectrum of a (1 + 1)D CFT with finite symmetry group G. An extension of this modular-invariance argument to 3 + 1 dimensions can be found in Reference 89.

The edge theory of the G_k CS theory is the G_k Wess–Zumino–Witten (WZW) model; it inherits the topological symmetry from the Wegner–Wilson lines running parallel to the boundary. These ingredients are used in Reference 88 to construct massless 2D quantum chromodynamics (QCD) with adjoint fermions by coupling CS theory on an interval to 2D Yang–Mills theory; the construction makes manifest some surprising noninvertible symmetries of the theory, which guarantee deconfinement.

In Reference 86 and 96, the authors argue that a (1 + 1)D system with fusion category symmetry can always be realized as a boundary condition of a gapped (2 + 1)D topological order with anyon types carrying the associated labels of the topological operators. They wish to study anomalies of the fusion category symmetry, to use them as RG invariants, and to label SPTs protected by such a symmetry: They are thinking of the bulk as a realization of anomaly inflow. Gapped edge theories are realized if the bulk theory admits gapped boundary conditions; such a bulk theory has an exactly solvable description as a string-net model (28). Explicit lattice models for gapped phases in $D = 1 + 1$ with fusion category symmetries appear in Reference 97.

Examples of systems with categorical symmetries include the anyon chain models studied in Reference 98, which uses the categorical symmetry to explain the gaplessness of the model. References 99–101 build classical lattice models whose defects realize a fusion category.

The terms categorical symmetry and noninvertible symmetry are not used in a unique way in the literature. In References 20, 102, and 103, the terms are used in the context of gapped phases in $D = 2 + 1$ with gapless boundaries; the idea is that such edge theories can have anomalies that go beyond those associated with invertible phases, which are therefore called noninvertible anomalies. The term algebraic higher symmetry is used in References 102 and 103 for the concept I called categorical symmetry above. References 102–104 propose that the most general notion of symmetry of a D-dimensional system is labeled by a topological order in one higher dimension.

6. GAPLESS STATES

6.1. Critical Points

The second part of the Landau paradigm (assertion 2) says that at a critical point, the critical degrees of freedom are the fluctuations of an order parameter. Apparent exceptions to this statement come in several varieties.

First, any transition out of a phase without a local order parameter presents an immediate problem. Consider the case of $\mathbb{Z}_2$ gauge theory in $D = 2 + 1$, which spontaneously breaks a $\mathbb{Z}_2$ one-form symmetry, with a charged loop operator $W[C]$. Can we understand the critical theory in terms of such a string order parameter field? By Wegner's duality (2), the local physics of the critical theory is in the same universality class as the 3D Ising model. This is yet another point of view from which the 3D Ising model should have a string theory dual (19, 105).

Second, there are direct transitions between states that break different symmetries, known as deconfined quantum critical points (DQCP; Reference 106 has a useful summary and references). Does this require a revision to assertion 2 of the Landau paradigm as stated above? There is a sense in which the degrees of freedom of the critical theory are simply the order parameters of both of the neighboring phases, coupled by a WZW term (107, 108). The presence of the WZW term is required by a mixed anomaly between the two symmetries. It says that defects of the order in one phase carry charge under the other (109). This perspective predicts a dramatic enlargement of symmetry at the critical point, which is not obvious from other points of view, and is borne out by numerical work. This symmetry-based description as a nonlinear sigma model has the serious shortcoming that it is strongly coupled, but so is the more-familiar description in terms of abelian gauge theory.

Independent of the extended Landau paradigm, I should also mention that the study of order parameters for higher-form symmetry at various critical points has been instructive (20, 20, 110, 111). In particular, this study has provided independent evidence that the $(2+1)$D DQCP between the Néel and VBS (valence bond solid) phases is a weakly first-order transition (111).

6.2. Gapless Phases

Gapless phases are a wild frontier of our understanding, and we certainly do not have a symmetry-based (or any other) understanding of all possibilities at the moment. I limit myself to remarks on two illuminating examples.

First, I mention a set of exotic gapless fractonic phases that can be constructed by assembling layers of quantum Hall states. They can be described at low energies by an abelian CS theory with a nearly diagonal K matrix whose size grows with the number of layers (112–114). For some choices of K matrix, this represents a new class of gapped fracton phase, with irrational particle statistics and a large-order fusion group. For other choices of K matrix, the spectrum is gapless. Reference 115 shows that the gapless examples of such states can be understood in terms of weak symmetry breaking (116). This means that the charged operator that condenses is not a local but rather an extended operator, in this case extended along the direction of the stack of layers.

Second, among the list above of apparent exceptions to the Landau paradigm, it remains to discuss the Landau Fermi liquid. Reference 117 gives something like a symmetry-based understanding of both Fermi liquids and a large class of non-Fermi liquids (for a review of the latter, see Reference 118). First, we assume translation symmetry, so that we may speak about a well-defined Fermi surface in momentum space. The key ingredient is an emergent symmetry representing independent particle number conservation at each point on the Fermi surface. In $2+1$ dimensions, where the simplest Fermi surface is a circle, this is a loop group symmetry; that is, the symmetry transformation is a map from the circle to $\mathsf{U}(1)$. Such a loop-group symmetry emerges in the Landau theory, as well as in a large class of non-Fermi liquids obtained by coupling a Fermi surface to gapless modes. Reference 117 shows that a state with a fractional and continuously variable filling must have such a large symmetry. From this starting point, the authors develop an understanding of Luttinger's theorem as an anomaly of this loop symmetry. (A related anomaly-based perspective on Luttinger's theorem appears in Reference 119.) It shows that in a system with such a loop group symmetry, a literal Fermi arc, i.e., a boundary of the Fermi surface, would imply a violation of charge conservation: The Fermi surface must be the boundary of some region of the Brillouin zone.

7. CONCLUDING REMARKS

7.1. Topological Local Operators

What about the case of $(D-1)$-form symmetries in D spacetime dimensions? This means that there are local operators that are topological. This case is studied in References 13–15 and more generally in References 88, 120, and 121. The conclusion is that the Hilbert space of such a system is divided into superselection sectors with different values of the topological operators. An example in which this arises is in gauge theory in $D = 1 + 1$ without minimally charged matter, where sectors represent different values of the electric flux. Reference 121 considers what happens when the action is perturbed by such operators, which are always relevant. The perturbation changes the difference of the vacuum energies between different sectors.

7.2. Higher Groups

The concept of higher groups can be regarded as a natural extension of higher-form symmetry (see, e.g., Reference 122 for a broader mathematical perspective). For example, a two-group structure can be defined in a physical context as follows: it is a modification of the current algebra of a one-form symmetry and a zero-form symmetry, so that the zero-form gauge transformation acts nontrivially on the two-form background field B for the one-form symmetry:

$$A \to A + \mathrm{d}\lambda,\ B \to B + \kappa\lambda \mathrm{d}A, \qquad 31.$$

where A is the background one-form field for the zero-form symmetry, and κ can be regarded as a structure constant. This construction is closely related to the Green–Schwarz mechanism of anomaly cancellation: Suppose, for example, the effective action of a (1 + 1)D theory with the above ingredients has an anomalous variation $\delta_\lambda S = \int \kappa\lambda \frac{\mathrm{d}A}{2\pi}$ under a zero-form gauge transformation. Then the modified action $S - \int \frac{B}{2\pi}$ is invariant under the transformation in Equation 31. Though it has not yet explicitly played a role in the condensed matter literature to my knowledge, it appears in many places in QFT (e.g., 91, 123–125) and we can expect that it will be useful.

7.3. Other Applications

In the preceding discussion, we have focused on generalizations of notions of symmetry as applied to zero-temperature ground states of quantum matter. I should mention that these same generalized symmetries have a number of other applications:

- A new organizing principle for magnetohydrodynamics (126–129). More generally, many kinds of exotic hydrodynamics can be understood by applying the systematic logic of hydrodynamics to a system with generalized symmetries (see, for example, Reference 130).
- Reference 131 provides a nice example using both anomalies and generalized symmetries to understand the spectrum of Goldstone modes of the standard model in a magnetic field and suggests a realization of the same physics in Dirac semimetals.
- More generally, more symmetry means more possible anomalies and, therefore, new anomaly constraints on IR behavior of QFT. For example, a mixed anomaly between time-reversal symmetry and a one-form symmetry implies an LSMOH constraint on the ground state of Yang–Mills theory at $\theta = \pi$ (132–135). Work in this direction includes References 136–143 and many others (136–145).

7.4. Disorder

I have not spoken about systems with disorder. Even if we are generally interested in clean systems, it is important to ask about the stability of our statements to the introduction of disorder. In the case of zero-form symmetries, the Imry–Ma argument for stability of SSB proceeds by coupling the local order parameter to the disorder. Naively, the inability to write such a coupling corroborates our expectation that higher-form SSB is even more robust (65).

7.5. Dynamics

I have focused entirely on equilibrium phases of matter. Dynamics of quantum matter is a current frontier, in which, of course, symmetries continue to play a crucial role. A generalization of the notion of symmetry that has appeared in this context is the phenomenon of Hilbert space fragmentation: This is what happens when the algebra of operators that commute with each term of the Hamiltonian grows exponentially with system size (146; for systems with ordinary symmetries, this algebra grows only polynomially with system size).

7.6. Still Beyond Landau?

In this review, I've tried to motivate the following question: Does the enlarged Landau paradigm (including all generalizations of symmetries, and their anomalies) incorporate all equilibrium quantum phases of matter (and transitions between them) as consequences of symmetry? Even if the answer is no, I think it has already been a fruitful question. I close by enumerating some outstanding possible exceptions to even the most generous interpretation in hopes of encouraging some further thought in this direction.

- Symmetries can forbid all relevant operators that would lift gapless modes that are not Goldstones. An example is chiral symmetry in QCD, which forbids fermion masses. A condensed matter example is the Dirac spin liquid—a phase described by a CFT with no symmetric relevant operators.
- Above I argued that the DQCP between two distinct symmetry-breaking phases satisfies assertion 2 of the Landau paradigm because it admits a description in terms of a nonlinear sigma model whose fields are the order parameters of the two phases. Reference 147 generalizes this description to a sigma model on the Stiefel manifold, the coset space $\mathsf{SO}(N+4)/\mathsf{SO}(4)$. For $N=1$ this is the DQCP; for $N=2$, these authors give evidence that this is a description of a Dirac spin liquid in terms of only gauge-invariant variables. The case $N=2$ is called Stiefel liquid, and Reference 147 provides a candidate microscopic realization with the argument that it has no weakly coupled limit.
- An extremely interesting example of a claimed exception to assertion 2 of even the Generalized Landau Paradigm is provided by phase transitions described by IR-free gauge theory (148). The claim of Reference 148 is that $\mathsf{SU}(N)$ gauge theory with adjoint fermions (take $N=2$) has a $\mathbb{Z}_2$ symmetry and describes, as the fermion mass changes sign, a completely novel critical theory for the transition from the trivial phase to the ordinary SSB phase. The degrees of freedom of this theory certainly go beyond the fluctuations of the order parameter. Notice that for any nonzero mass there is an extra emergent one-form symmetry associated with the center of the gauge group. A physical consequence of this symmetry (and a mixed anomaly), were it exact, would be that a domain wall between the two $\mathbb{Z}_2$-breaking vacua would satisfy an LSMOH constraint; that is, the domain walls of the ordered phase would carry some extra degrees of freedom, and this would distinguish this phase from the ordinary SSB phase. This symmetry is, however, explicitly broken by the massive charged matter of the gauge theory.

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