ANNUAL REVIEWS

Annual Review of Control, Robotics, and Autonomous Systems

Port-Hamiltonian Modeling for Control

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Annu. Rev. Control Robot. Auton. Syst. 2020. 3:393–416

First published as a Review in Advance on October 10, 2019

The *Annual Review of Control, Robotics, and Autonomous Systems* is online at control.annualreviews.org

https://doi.org/10.1146/annurev-control-081219-092250

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Keywords

multiphysics systems, Hamiltonian systems, network modeling, Dirac structures, passivity, compositionality, control by interconnection, impedance, gradient systems, algebraic constraints

Abstract

This article provides a concise summary of the basic ideas and concepts in port-Hamiltonian systems theory and its use in analysis and control of complex multiphysics systems. It gives special attention to new and unexplored research directions and relations with other mathematical frameworks. Emergent control paradigms and open problems are indicated, including the relation with thermodynamics and the question of uniting the energy-processing view of control, as emphasized by port-Hamiltonian systems theory, with a complementary information-processing viewpoint.

1. INTRODUCTION

Port-based modeling of physical systems is based on viewing the system as the interconnection of three types of ideal elements: dynamical energy-storing elements, static energy-dissipating elements, and static lossless energy-routing elements. These ideal modeling constructs are interconnected by (vector) pairs of variables, which are conjugate in the sense that their product equals power (energy divided by time)—for example, pairs of generalized forces and velocities in the mechanical domain and pairs of currents and voltages in the electrical domain. Since energy is the lingua franca among different physical domains (mechanical, electrical, electromechanical, hydraulic, chemical, etc.), it provides a general framework for modeling multiphysics systems. This turns out to be an insightful and surprisingly powerful method for modeling, certainly for control purposes.

Port-based modeling has multiple origins, including electrical network theory, and was pioneered in the late 1950s by Paynter (1). During this period, there was widespread interest in unifying the modeling frameworks of different physical domains because of engineering needs to break borders between different disciplines (the rise of mechatronic systems, energy conversion in machines, etc.). Port-based modeling together with the accompanying graphical notation of bond graphs (or related object-oriented modeling languages) has served since then as an attractive option for systematic modeling of multiphysics systems for the purposes of simulation, control, and design (see 2–4).

The geometric, coordinate-free formalization of port-based modeling started in the early 1990s, leading to the notion of port-Hamiltonian systems (3, 5–10). The key idea is that the totality of the energy-routing elements and interconnection topology of the system (the generalized junction structure, in bond graph terminology) defines a geometric structure, commonly known as a Dirac structure. From a geometric mechanics point of view, Dirac structures generalize both symplectic and Poisson structures (11, 12). In combination with the Hamiltonian specified by the energy-storing elements, the dynamics is given in Hamiltonian form with respect to the Dirac structure, with the addition of energy-dissipating and external (interaction) ports. Thus, the geometric structure of port-Hamiltonian systems is based on the interconnection (network) structure, in contrast to the symplectic or Poisson structure of classical Hamiltonian systems, which is based primarily on the geometry of the cotangent bundle of the configuration manifold (13–16).

Thus, from a historical perspective, port-Hamiltonian systems theory combines classical network theory and the general systems point of view with geometry, thereby providing a synthesis between multiphysics network systems and the geometric formulation of dynamical systems.

2. FROM PORT-BASED MODELING TO PORT-HAMILTONIAN SYSTEMS

The essence of port-based modeling and port-Hamiltonian systems is represented in **Figure 1**. The energy-storing elements $\mathcal{S}$ and the energy-dissipating elements $\mathcal{R}$ are linked to a central energy-routing structure, geometrically defined as a Dirac structure. This linking takes place via pairs (f, e) of equally dimensioned vectors f and e (commonly called flow and effort variables, although we will not attach any special meaning to this terminology). A pair (f, e) of vectors of flow and effort variables defines a port, and the total set of variables f, e is also called the set of port variables.

Figure 1 shows three ports: the port (f_S, e_S) linking to energy storage, the port (f_R, e_R) corresponding to energy dissipation, and the external port (f_P, e_P), by which the system interacts with its environment (including controller action). The scalar quantities $e_S^T f_S$, $e_R^T f_R$, and $e_P^T f_P$ denote the instantaneous powers transmitted through the links (the bonds, in bond graph terminology).

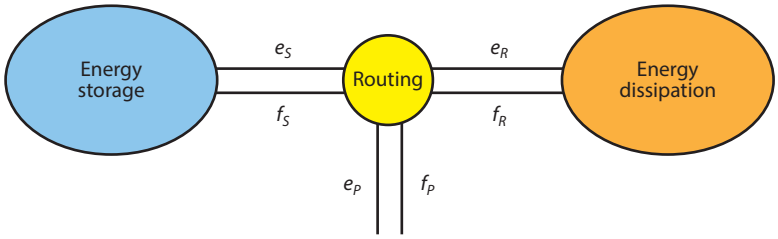

Figure 1

From port-based modeling to a port-Hamiltonian system.

Any physical system that is represented (modeled) in this way defines a port-Hamiltonian system. Furthermore, experience has shown that, even for very complex physical systems, port-based modeling leads to satisfactory and insightful models, certainly for control purposes (modeling for control) (see, e.g., 2, 3, 8, 17, and references therein).

2.1. Port-Hamiltonian Systems

In this section, we discuss the three types of building blocks of port-Hamiltonian systems theory in more detail.

2.1.1. Energy routing and Dirac structures. The geometric notion of a Dirac structure captures the energy-routing elements and the interconnection topology of port-Hamiltonian systems. In electrical network terminology, the Dirac structure is the printed circuit board (without the energy-storing and energy-dissipating components) and thus provides the wiring for the overall system.

The basic property of a Dirac structure is power conservation: The Dirac structure links the flow and effort variables $f = (f_S, f_R, f_P)$ and $e = (e_S, e_R, e_P)$ in such a way that the total power $e^T f$ is equal to zero. In the formal definition of a Dirac structure, we start with a finite-dimensional linear space of flows $\mathcal{F}$ and the dual linear space of efforts $\mathcal{E} := \mathcal{F}^*$.

Remark 1. Usually, one takes $\mathcal{F} = \mathbb{R}^k$ and $\mathcal{E} = \mathbb{R}^k$. However, in some cases (such as rigid-body dynamics), $\mathcal{F}$ is an abstract linear space, such as the space of twists $\mathcal{F} = \mathrm{se}(3)$, the Lie algebra of the matrix group SE(3), with the space of efforts $\mathcal{E}$ given as the linear space of wrenches $\mathcal{E} = \mathrm{se}^*(3)$, the dual of the Lie algebra se(3). In the infinite-dimensional case (see, e.g., 18), even more care should be taken.

The power P on the total space $\mathcal{F} \times \mathcal{E}$ of port variables is defined by the duality product $P = < e \mid f >$. In the common case $\mathcal{F} = \mathcal{E} = \mathbb{R}^k$, this simply amounts to $P = e^T f$.

Definition 1 (from References 11 and 12). Consider a finite-dimensional linear space $\mathcal{F}$ with $\mathcal{E} = \mathcal{F}^*$. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure if (*a*) $< e \mid f > = 0$ for all $(f, e) \in \mathcal{D}$ and (*b*) $\dim \mathcal{D} = \dim \mathcal{F}$.

It can be seen that the maximal dimension of any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ satisfying the power-conservation property (item *a* in Definition 1) is equal to $\dim \mathcal{F}$. Thus, a Dirac structure is a maximal power-conserving subspace.

2.1.2. Energy storage and energy dissipation. Energy storage introduces dynamics. Let f_S, e_S be the vector of flow and effort variables of the energy-storage port. Integrating the flow variables

$-f_S$ leads to the equally dimensioned vector of state variables $x \in \mathcal{X}$ satisfying $\dot{x} = -f_S$. Energy storage is expressed by the Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, defining the vector e_S of effort variables as $e_S = \nabla H(x)$, where $\nabla H(x)$ is the column vector of partial derivatives of H. Obviously, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = (\nabla H(x(t)))^T \dot{x}(t) = -e_S^T(t) f_S(t). \qquad 1.$$

Energy dissipation is any relation $\mathcal{R}$ between the flow and effort variables f_R, e_R of the energy-dissipating port such that

$$e_R^T f_R \leq 0, \quad (f_R, e_R) \in \mathcal{R}. \qquad 2.$$

Consider now a Dirac structure

$$\mathcal{D} \subset \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P \times \mathcal{E}_S \times \mathcal{E}_R \times \mathcal{E}_P \qquad 3.$$

together with energy storage defined by $H : \mathcal{X} \to \mathbb{R}$ and an energy-dissipation relation $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$. Then the resulting port-Hamiltonian system is geometrically defined as the implicit dynamics

$$(-\dot{x}(t), f_R(t), f_P(t), \nabla H(x(t)), e_R(t), e_P(t)) \in \mathcal{D}, \quad (f_R(t), e_R(t)) \in \mathcal{R},\ t \in \mathbb{R}, \qquad 4.$$

in the state variables x, with external port variables f_P, e_P. Using one of the various ways (7, 9, 10) to represent Dirac structures by equations, this typically results in a mixture of differential and algebraic equations.

A more specific class of port-Hamiltonian systems is obtained as follows. A key example of a Dirac structure is the graph of any skew-symmetric map from $\mathcal{E}$ to $\mathcal{F}$. For instance, let the Dirac structure $\mathcal{D}$ be given as the graph of the skew-symmetric map

$$\begin{bmatrix} -J & -G_R & -G \\ G_R^T & 0 & 0 \\ G & 0 & 0 \end{bmatrix}, \quad J = -J^T,$$

from e_S, e_R, e_P to f_S, f_R, f_P. Furthermore, let energy dissipation be given by the linear relation $e_R = -\bar{R} f_R$ for some matrix $\bar{R} = \bar{R}^T \geq 0$. This yields the input–state–output port-Hamiltonian system

$$\begin{aligned} \dot{x} &= [J - R]\nabla H(x) + Gu, \\ y &= G^T \nabla H(x), \end{aligned} \qquad 5.$$

where $u = e_P$ is the input vector, $y = f_P$ is the output vector, and $R := G_R \bar{R} G_R^T \geq 0$.

In modeling, a large part of the Dirac structure is often determined by underlying balance laws. For example, in the case of electrical networks, the Dirac structure is defined (apart from possible transformers) by the combination of Kirchhoff's current and voltage laws relating the currents through and voltages across the edges of the network graph. Furthermore, in a mass–spring–damper system, the Dirac structure is determined by the incidence matrix D of the directed graph, with nodes representing the masses and edges corresponding to springs and dampers, together with a matrix E whose columns correspond to the externally actuated masses. Thus, D is equal to $[D_\mathrm{s} \quad D_\mathrm{d}]$, where D_s is the spring incidence matrix and D_d is the damper incidence matrix. The

dynamics takes the port-Hamiltonian form (see 19)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \left(\begin{bmatrix} 0 & D_{\mathrm{s}}^T \\ -D_{\mathrm{s}} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D_{\mathrm{d}} \bar{R} D_{\mathrm{d}}^T \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} F,$$
$$v = E^T \frac{\partial H}{\partial q}(q,p), \qquad 6.$$

where $\bar{R}$ is a positive diagonal matrix of damping coefficients, F is the vector of external forces, and v is the vector of velocities of the actuated masses. In the infinite-dimensional case, the Stokes–Dirac structure (18) has an analogous structure, with the incidence matrix D replaced by the exterior derivative.

In quite a few cases of interest (e.g., 3-D mechanical systems), the above definition of a port-Hamiltonian system is not general enough, and we need to generalize the definition of a Dirac structure to a Dirac structure on a state-space manifold $\mathcal{X}$. Basically, this means that for every $x \in \mathcal{X}$,

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_P \times T_x^*\mathcal{X} \times \mathcal{E}_R \times \mathcal{E}_P,$$

i.e., the linear space of flows and efforts as specified by the Dirac structure, is modulated by the state x. In the special case of Equation 5, this means that the matrices J, R, and G may depend on x. (For more information, see, e.g., References 7, 8, and 11.)

2.2. Basic Properties of Port-Hamiltonian Systems

Port-Hamiltonian systems enjoy a number of structural properties that can be fruitfully used for analysis and control.

2.2.1. Passivity. A key property of any port-Hamiltonian system is the following. Combining the power-preservation property $e_S^T f_S + e_R^T f_R + e_P^T f_P = 0$ of any Dirac structure with the energy-storage property given by Equation 1 and the energy-dissipation relation in Equation 2 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = e_R^T(t) f_R(t) + e_P^T(t) f_P(t) \leq e_P^T(t) f_P(t). \qquad 7.$$

That is, an increase in stored energy H is less than or equal to the externally supplied power. If H is bounded from below, then this means that the port-Hamiltonian system is passive with respect to the supply rate $e_P^T f_P$ and storage function H. Actually, in the linear case, the converse can also be shown (9, 10), in the sense that any passive system with a quadratic storage function $x^T Q x$ with $Q > 0$ can be written as a port-Hamiltonian system with Hamiltonian $x^T Q x$ for some $J = -J^T$ and $R = R^T \geq 0$. Note, however, that the matrices J and R in the port-Hamiltonian formulation obtained from port-based modeling have a direct physical meaning. Converse results in the nonlinear case are more subtle (8). In general, port-Hamiltonian systems are much more structured than passive systems due to the explicit separation of energy storage, energy routing, and energy dissipation.

2.2.2. Shifted passivity. Passivity is especially useful for the stability analysis of the zero state of the system, corresponding to zero input and a minimum of the Hamiltonian H. On the other hand, in the case of dynamical distribution networks, as exemplified by power networks or chemical reaction networks in systems biology, the stability scenario is different. These systems normally operate under nonzero environmental conditions, such as nonzero generated and consumed power

in power networks and nonzero external inflow and outflow of chemical species in metabolic pathways in systems biology. Thus, a key stability question in dynamical distribution networks concerns the stability of the steady state for constant nonzero input (out of equilibrium).

In the case of a constant Dirac structure, such as Equation 5 for constant J, R, and G, we can proceed as follows (for the general constant Dirac structure case, see 8). Consider any constant $\bar{u}$ with a corresponding steady state $\bar{x}$, i.e.,

$$0 = [J - R]\,\frac{\partial H}{\partial x}(\bar{x}) + G\bar{u}, \quad \bar{y} = G^T\,\frac{\partial H}{\partial x}(\bar{x}).$$

Then Equation 5 can be rewritten as

$$\begin{aligned} \dot{x} &= [J - R]\,\frac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x) + G(u - \bar{u}), \\ y - \bar{y} &= G^T\,\frac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x), \end{aligned} \qquad 8.$$

where

$$\widehat{H}_{\bar{x}}(x) := H(x) - \frac{\partial H}{\partial x^T}(\bar{x})(x - \bar{x}) - H(\bar{x}) \qquad 9.$$

is the shifted Hamiltonian (also called the Bregman divergence in convex analysis). By additionally assuming that H is convex in a neighborhood of $\bar{x}$, one can see that the shifted Hamiltonian $\widehat{H}_{\bar{x}}$ has a minimum at $\bar{x}$. In this case, Equation 5 is dissipative with respect to the shifted supply rate $(y - \bar{y})^T(u - \bar{u})$, with storage function $\widehat{H}_{\bar{x}}$, and is called shifted passive.

Example 1 (stability of the swing equation model of a power network). The swing equation model of a power network is given by the port-Hamiltonian system (20)

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega, \\ y &= \frac{\partial H}{\partial p}(q,p) = \omega, \end{aligned} \qquad 10.$$

where p and ω denote the momentum and frequency, respectively, deviations corresponding to the synchronous machines at the nodes; q denotes the angle differences across every edge of the power network, whose topology is described by the incidence matrix D; and A denotes a positive diagonal matrix of damping constants. Furthermore, u is the vector of generated or consumed power at the nodes, and the Hamiltonian is of the form

$$H(q,p) = \frac{1}{2}p^T M^{-1} p - \sum \gamma_i \cos q_i, \qquad 11.$$

with constant γ_i determined by the physical properties of the ith transmission line and the voltages at its adjacent nodes (which are assumed to be constant).

Let $\bar{u}$ be a constant input, yielding steady-state values $(\bar{q}, \bar{p} = M\bar{\omega})$. Then $D^T\bar{\omega}$ is equal to zero, and thus, if the network is connected, $\bar{\omega}$ is equal to $\mathbb{1}\omega_*$ for some common frequency deviation ω_*, with $\mathbb{1}$ denoting the vector of all ones. Furthermore,

$$D\Gamma\,\mathrm{Sin}\,\bar{q} = -A\mathbb{1}\omega_* + \bar{u}, \qquad 12.$$

where Γ is the diagonal matrix with diagonal elements γ_i, and Sin denotes the element-wise sine function. Premultiplying both sides by $\mathbb{1}^T$ yields $\mathbb{1}^T A \mathbb{1} \omega_* = \mathbb{1}^T \bar{u}$, implying that $\omega_* = 0$ (frequency regulation) if and only if $\mathbb{1}^T \bar{u} = 0$ (generated power = consumed power). The shifted Hamiltonian is

$$\widehat{H}_{(\bar{q},\bar{p})}(q,p) := \frac{1}{2}(p-\bar{p})^T M^{-1}(p-\bar{p}) - \sum \gamma_i \left[\cos q_i + (q_i - \bar{q}_i)\sin \bar{q}_i - \cos \bar{q}_i\right], \qquad 13.$$

which, by strict convexity of H in p and $q \in (-\frac{\pi}{2}, \frac{\pi}{2})^n$, has a strict minimum at $(\bar{q}, \bar{p})$ whenever $\bar{q} \in (-\frac{\pi}{2}, \frac{\pi}{2})^n$. In particular, for $u = \bar{u}$, this implies that the steady state $(\bar{q}, \bar{p})$ is asymptotically stable.

2.2.3. Casimirs of port-Hamiltonian systems. As in classical Hamiltonian systems, there is a rich theory concerning symmetries and conserved quantities for port-Hamiltonian systems (see, e.g., 21, 22). Let us concentrate on the conserved quantities of the port-Hamiltonian systems, as in Equation 5 for $u = 0$, with J and R possibly depending on the state x. In particular, let us look at conserved quantities $C : \mathcal{X} \to \mathbb{R}$ that are independent of the Hamiltonian H. They are determined as solutions of the partial differential equations

$$\frac{\partial^T C}{\partial x}(x)[J(x) - R(x)] = 0, \qquad 14.$$

implying that $\frac{\mathrm{d}}{\mathrm{d}t} C = 0$ for $u = 0$. Such conserved quantities are called Casimirs of the system for $u = 0$ [extending the classical definition in the case $R(x) = 0$]. Postmultiplying by $\frac{\partial C}{\partial x}(x)$ and using the skew symmetry of $J(x)$ and positive semidefiniteness of $R(x)$, one can easily see that Equation 14 is equivalent to

$$\frac{\partial^T C}{\partial x}(x) J(x) = 0, \quad \frac{\partial^T C}{\partial x}(x) R(x) = 0. \qquad 15.$$

Casimirs play a major role in stability analysis, since any nonlinear combination $\Phi(H, C) : \mathcal{X} \to \mathbb{R}$ of H and C, with $\Phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\frac{\partial \Phi}{\partial z_1}(H(x), C(x)) \geq 0$, satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(H, C) \leq 0$ and thus is a candidate Lyapunov function. In classical mechanics, this approach (for $R = 0$) is commonly known as the energy-Casimir method (13, 14). It also turns out to be a key element in set-point stabilization by control by interconnection, as discussed in the next section.

2.2.4. Compositionality. Inherent to the definition of port-Hamiltonian systems is the fact that the interconnection of port-Hamiltonian systems through any interconnection Dirac structure is again a port-Hamiltonian system. In fact, the Hamiltonian H of the interconnection of N port-Hamiltonian systems is simply the sum $H = H_1 + \cdots + H_N$ of the Hamiltonians H_j, $j = 1, \ldots, N$ of the subsystems. Furthermore, the energy dissipation is just the collection of the energy-dissipating parts of the subsystems. On the other hand, the Dirac structure of the interconnected system is defined as the composition of the Dirac structures $\mathcal{D}_j$ of the individual subsystems, together with the interconnection Dirac structure. This is based on the following key fact. Consider two Dirac structures $\mathcal{D}_1, \mathcal{D}_2$, involving the flows and efforts $f_i, f_i^c, e_i, e_i^c, i = 1, 2$. Here, $f_i^c, e_i^c, i = 1, 2$, are the flow and effort variables to be connected, living in the same space of flows and efforts. Then define the composition

$$\mathcal{D}_1 \circ \mathcal{D}_2 := \{(f_1, e_1, f_2, e_2) \mid \exists f_1^c, e_1^c \text{ such that (s.t.) } (f_1, f_1^c, e_1, e_1^c) \in \mathcal{D}_1, (f_2, -f_1^c, e_2, e_1^c) \in \mathcal{D}_2\}, \quad 16.$$

corresponding to the interconnection $f_2^c = -f_1^c, e_2^c = e_1^c$. It can be shown (23) that the subspace $\mathcal{D}_1 \circ \mathcal{D}_2$ of flows f_1, f_2 and efforts e_1, e_2 is again a Dirac structure. This result immediately extends to the composition of multiple Dirac structures.

The compositionality of port-Hamiltonian systems can be regarded as a (far-reaching) generalization of the classical passivity theorem (8, 24), stating that the negative feedback interconnection $u_1 = -y_2 + v_1, u_2 = y_1 + v_2$ of two passive systems with inputs u_i and outputs $y_i, i = 1, 2$, is passive with respect to the inputs v_1, v_2 and outputs y_1, y_2. (Note in particular the power-conservation property $y_1^T u_1 + y_2^T u_2 = y_1^T v_1 + y_2^T v_2$ of the negative feedback interconnection.)

3. CONTROL OF PORT-HAMILTONIAN SYSTEMS

Port-based modeling not only provides a systematic and compositional framework for first-principles modeling of multiphysics systems, but also identifies the underlying physical structure in the obtained mathematical models. Some key properties for the analysis of port-Hamiltonian systems were discussed in the previous section. In this section, we discuss the use of port-Hamiltonian structure for control, aimed at robust and physically interpretable control strategies.

For concreteness, we confine ourselves to port-Hamiltonian systems of the standard form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\nabla H(x) + G(x)u, \\ y &= G^T(x)\nabla H(x), \end{aligned} \qquad 17.$$

although most ideas and many results can be extended to more general situations, such as nonlinear energy dissipation, the presence of algebraic constraints, or even infinite-dimensional (distributed-parameter) systems.

3.1. Control by Interconnection

A powerful paradigm for the control of port-Hamiltonian systems is control by interconnection, where we consider controller systems that are also port-Hamiltonian, and shape the dynamics of the given plant port-Hamiltonian system to a desired closed-loop port-Hamiltonian dynamics.

3.1.1. Set-point stabilization. Consider the port-Hamiltonian system as in Equation 17. The simplest control problem is set-point stabilization, where we aim at designing a controller such that the plant state of the closed-loop system converges to a given desired set-point value x^*. The easiest case is when the set point x^* is a strict minimum of the Hamiltonian H. Indeed, this means that x^* is already a stable equilibrium of the uncontrolled ($u = 0$) port-Hamiltonian system. Applying negative output feedback $u = -y$ results in

$$\frac{\mathrm{d}}{\mathrm{d}t}H = -(\nabla H(x))^T \left[R(x) + G(x)G^T(x)\right]\nabla H(x) \leq 0, \qquad 18.$$

and asymptotic stability can be investigated with the help of LaSalle's invariance principle.

Now consider the case that x^* is not a strict minimum of H. If x^* is a steady state corresponding to a constant input u^* and J, R, and G are all constant—i.e., $[J - R]\nabla H(x^*) + Gu^* = 0$—then the stability of x^* for $u = u^*$ may be investigated using the shifted Hamiltonian $\widehat{H}_{x^*}$, as discussed above. In particular, if H is strictly convex, then $\widehat{H}_{x^*}$ has a strict minimum at x^*, implying stability, while asymptotic stability may be pursued by additional output feedback—i.e., $u = u^* - (y - y^*)$, where y^* is the steady-state output value.

Next, let us consider the case that x^* is an equilibrium of Equation 17 but not a strict minimum of H. In this case, one option is to use the Casimirs of the system, as introduced in Section 2.2.3.

The reason is that, as noted above, any (nonlinear) combination $\Phi(H, C) : \mathcal{X} \to \mathbb{R}$ of H and C, with $\Phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying $\frac{\partial \Phi}{\partial z_1}(H(x), C(x)) \geq 0$, satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(H, C) \leq 0$ and thus defines a candidate Lyapunov function. Importantly, the minimum of $\Phi(H, C)$ may be different from the minimum of H, and thus x^* can be a strict minimum of this newly created Lyapunov function candidate. If C and Φ are found such that $V := \Phi(H, C)$ has a strict minimum at x^*, then stability for $u = 0$ results, while asymptotic stabilization can be pursued by adding negative output feedback with respect to the shaped output $\widetilde{y} = G^T(x)\nabla V(x)$. This energy-Casimir method (for $R = 0$) was used classically to analyze the stability of the nonzero equilibria of Euler's equations for the angular velocity dynamics of a rigid body.

What can we do if this all fails? The next option is to consider dynamical controller systems, also given as port-Hamiltonian systems,

$$\begin{aligned} \dot{\xi} &= [J_c(\xi) - R_c(\xi)]\,\nabla H_c(\xi) + G_c(\xi)u_c, \quad \xi \in \mathcal{X}_c, \\ y_c &= G^T(\xi)\nabla H_c(\xi), \end{aligned} \tag{19.}$$

via standard negative feedback $u = -y_c, u_c = y$. By compositionality, the closed-loop system is the port-Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \left(\begin{bmatrix} J(x) & -G(x)G_c^T(\xi) \\ G_c(\xi)G^T(x) & J_c(\xi) \end{bmatrix} - \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \xi}(\xi) \end{bmatrix}, \tag{20.}$$

with state space $\mathcal{X} \times \mathcal{X}_c$ and total Hamiltonian $H(x) + H_c(\xi)$. At first sight, this does not seem to help, since the dependency of H on x is not changed. The idea, however, is to design the port-Hamiltonian controller system in such a manner that the closed-loop system has useful Casimirs $C(x, \xi)$, leading to candidate Lyapunov functions

$$V(x, \xi) := \Phi(H(x), H_c(\xi), C(x, \xi)), \tag{21.}$$

with Φ satisfying $\frac{\partial \Phi}{\partial z_1} \geq 0$, $\frac{\partial \Phi}{\partial z_2} \geq 0$. Indeed, the strategy is to generate Casimirs $C(x, \xi)$ whose x dependency is used to shape the x dependency of V in a desirable way, where the still-to-be-determined function $H_c(\xi)$ can be used to shape the ξ dependency of V. Using the theory of Casimirs as exposed in Section 2.2.3, we therefore look for functions $C(x, \xi)$ satisfying

$$\begin{aligned} &\begin{bmatrix} \frac{\partial^T C}{\partial x}(x, \xi) & \frac{\partial^T C}{\partial \xi}(x, \xi) \end{bmatrix} \begin{bmatrix} J(x) & -G(x)G_c^T(\xi) \\ G_c(\xi)G^T(x) & J_c(\xi) \end{bmatrix} = 0, \\ &\begin{bmatrix} \frac{\partial^T C}{\partial x}(x, \xi) & \frac{\partial^T C}{\partial \xi}(x, \xi) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} = 0, \end{aligned} \tag{22.}$$

such that V has a minimum at (x^*, ξ^*) for some (or a set of) ξ^*. This already implies that the set point x^* is stable. In order to obtain asymptotic stability, one extends the negative feedback $u = -y_c,\ u_c = y$ by including extra damping,

$$u = -y_c - G^T(x)\frac{\partial V}{\partial x}(x, \xi), \quad u_c = y - G_c^T(x)\frac{\partial V}{\partial \xi}(x, \xi), \tag{23.}$$

and asymptotic stability is investigated through the use of LaSalle's invariance principle. This control scheme has been successfully used in a number of applications (see, e.g., 8, 25–27, and the references therein).

A somewhat unexpected consequence of the second line of Equation 22 is that $\frac{\partial^T C}{\partial x}(x,\xi)R(x) = 0$, implying that the presence of energy dissipation in the plant system places severe restrictions on the existence of Casimirs for the closed-loop system, and thus on the possibility of shaping V in a desirable way. This is referred to as the dissipation obstacle. While in the context of mechanical systems the dissipation obstacle is not a real obstacle (since energy dissipation appears in the differential equations for the momenta, while the kinetic energy does not need to be shaped), it does play a major role in other cases.

Various ways of overcoming the dissipation obstacle have been investigated. The most prominent is to look for alternate outputs for the plant system such that the system is still port-Hamiltonian with respect to this new output. Consider instead of the given output $y = G^T(x)\nabla H(x)$ any other output

$$y_A := [G'(x) + P(x)]^T \nabla H(x) + [M(x) + S(x)]u \qquad 24.$$

for G', P, M, and S satisfying

$$G(x) = G'(x) - P(x), \quad M(x) = -M^T(x), \quad \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix} \geq 0. \qquad 25.$$

Any such alternate output still satisfies $\frac{\mathrm{d}}{\mathrm{d}t}H \leq u^T y_A$ and defines a port-Hamiltonian system (of a slightly more general form than in Equation 17). (For a summary of the developed theory and additional references, see Reference 8.)

The search for Casimirs of the closed-loop port-Hamiltonian system also has an interesting state-feedback interpretation. For concreteness, consider Casimirs of the form $C_i(x,\xi) := \xi_i - F_i(x)$, $i = 1,\ldots,n_c$, with n_c the dimension of the port-Hamiltonian controller system. Since the Casimirs are constant along trajectories of the closed-loop system, it follows that in this case the controller states ξ can be expressed as $\xi_i = F_i(x) + \lambda_i, i = 1,\ldots,n_c$, for constants λ_i depending on the initial conditions. This defines a foliation of invariant manifolds L_λ of the closed-loop system, on each of which the dynamics is given as

$$\dot{x} = [J(x) - R(x)]\,\frac{\partial H_s}{\partial x}(x), \qquad 26.$$

with shaped Hamiltonian $H_s(x) := H(x) + H_c(F(x) + \lambda)$. On the other hand, this dynamics could have been obtained directly by applying the state feedback

$$\alpha_\lambda(x) = -G_c^T(F(x) + \lambda)\frac{\partial H_c}{\partial \xi}(F(x) + \lambda). \qquad 27.$$

The next option in this state-feedback approach is to add other degrees of freedom for obtaining a suitable shaped H_s by searching for state feedbacks $u = \alpha(x)$ such that

$$[J(x) - R(x)]\nabla H(x) + G(x)\alpha(x) = [J_s(x) - R_s(x)]\nabla H_s(x), \qquad 28.$$

where $J_s(x) = -J_s^T(x)$ and $R_s(x) = R_s^T(x) \geq 0$ are to be newly assigned. This is called interconnection-damping-assignment passivity-based control (see 8, 25–27). Note that, assuming that $G(x)$ has full column rank, the solvability of Equation 28 in terms of H_s and α is equivalent to solving

$$G^\perp(x)[J(x) - R(x)]\nabla H(x) = G^\perp(x)[J_s(x) - R_s(x)]\nabla H_s(x) \qquad 29.$$

in terms of H_s only, where $G^\perp(x)$ is a full-rank annihilator of $G(x)$.

3.1.2. Impedance control. Consider a port-Hamiltonian system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + G(x)u + L(x)f, \quad x \in \mathcal{X}, \\ y &= G^T(x)\frac{\partial H}{\partial x}(x), \quad u, y \in \mathbb{R}^m, \\ e &= L^T(x)\frac{\partial H}{\partial x}(x), \quad f, e \in \mathbb{R}^m, \end{aligned} \tag{30.}$$

where we distinguish between two (multidimensional) ports: the control port with conjugate variables u, y, and the external port f, e. Examples can be found in robotics, where u, y is the actuation port and f, e is the interaction port. In such a context, the relation between f and e (velocity and force, respectively) is often referred to as the impedance (see, e.g., 28). An important control goal is to achieve a desired impedance (e.g., for pick-and-place operations) through the use of the control port u, y. Thus, the questions arise of characterizing the achievable impedances and how to optimize among the achievable ones (depending on the tasks to be performed).

Despite its importance, this problem has not been studied in any mathematical depth. Let us instead look at the related static stiffness control problem. Consider a network of linear springs with two ports: the interaction port f, e, and the control port u, y, but with f, u now denoting displacements and e, y denoting the corresponding (spring) forces. The equations are given as

$$\begin{bmatrix} e \\ y \end{bmatrix} = K \begin{bmatrix} f \\ u \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} f \\ u \end{bmatrix}, \tag{31.}$$

where the stiffness matrix K satisfies $K = K^T \geq 0$. Applying extra springs with stiffness matrix K_c at the control port results in

$$\begin{bmatrix} e \\ 0 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} + K_c \end{bmatrix} \begin{bmatrix} f \\ u \end{bmatrix}, \tag{32.}$$

yielding the effective stiffness at the interaction port given as

$$e = K_{\text{res}} f := \left(K_{11} - K_{12} \left(K_{22} + K_c \right)^{-1} K_{21} \right) f. \tag{33.}$$

Translating the results on achievable behavior of resistive electrical networks as obtained in Reference 29 to the present case immediately yields the following characterization of the achievable effective stiffness.

Proposition 1. An effective stiffness $e = K_{\text{res}} f$ is achievable by proper choice of $K_c = K_c^T \geq 0$ if and only if (*a*) $K_{\text{res}}|_{\ker K_{21}} = K_{11}|_{\ker K_{21}}$ and (*b*) $K_{\text{res}} \geq K_{11} - K_{12} K_{22}^{-1} K_{21}$.

3.2. Energy-Routing Control

An interesting energy-efficient option for control is to directly influence the energy-routing part of the port-Hamiltonian system. This can be pursued by controlling the Dirac structure $\mathcal{D}$, which in mechanical systems is referred to as variable transmission. Another option is to modify the Dirac structure by an additional energy router, as exemplified in the following scenario. Consider two port-Hamiltonian systems,

$$\begin{aligned} \dot{x}_i &= J_i(x_i)\frac{\partial H_i}{\partial x_i}(x_i) + G_i(x_i)u_i, \\ y_i &= G_i^T(x_i)\frac{\partial H_i}{\partial x_i}(x_i), \quad i = 1, 2, \end{aligned} \tag{34.}$$

where we want to transfer energy from system 1 to system 2 while keeping the total energy $H_1 + H_2$ constant. Apply the nonlinear output feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \qquad 35.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_1 = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0, \qquad 36.$$

implying that H_1 is decreasing as long as $||y_1||$ and $||y_2||$ are different from 0. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t} H_2 = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0, \qquad 37.$$

implying that H_2 is increasing at the same rate. Hence, the output feedback acts as an energy router that irreversibly transfers all the energy of system 1 to system 2. This particular control scheme was successfully applied to energy-efficient path-following control of mechanical systems in Reference 30. For other control strategies explicitly based on the port-Hamiltonian structure of the system, readers can refer to References 8 and 31.

4. EXTENSIONS OF PORT-HAMILTONIAN SYSTEMS

In this section, we discuss some extensions to the definition of port-Hamiltonian systems and their relationships to other classes of systems, and we indicate the potential of these extensions for control purposes.

The starting point of the definition of a general port-Hamiltonian system is a Dirac structure $\mathcal{D} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$, where the flow and effort variables $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$ are terminated by an energy-dissipating relation $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ satisfying $e_R^T f_R \leq 0$ for all $(f_R, e_R) \in \mathcal{R}$. It follows that the composition of $\mathcal{D}$ with $\mathcal{R}$, defined as

$$\mathcal{D} \circ \mathcal{R} := \{(f_S, e_S, f_P, e_P) \mid \exists (f_R, e_R) \in \mathcal{R} \text{ s.t. } (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{D}\},$$

satisfies the property $e_S^T f_S + e_P^T f_P = -e_R^T f_R \geq 0$ for all $(f_S, e_S, f_P, e_P) \in \mathcal{D} \circ \mathcal{R}$. Hence, a more general viewpoint on port-Hamiltonian systems is not to distinguish between $\mathcal{D}$ and $\mathcal{R}$ but instead to start from a general nonlinear relation,

$$\mathcal{N} := \{(f_S, e_S, f_P, e_P) \in \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P \mid e_S^T f_S + e_P^T f_P \geq 0\}, \qquad 38.$$

combining the Dirac structure $\mathcal{D}$ and the energy-dissipating relation $\mathcal{R}$ into a single object.

This leads to two related new directions. The first is the theory of incrementally port-Hamiltonian systems. The second is the connection of port-Hamiltonian systems with pseudo-gradient systems (generalizing the Brayton–Moser equations for electrical circuits).

4.1. Incrementally Port-Hamiltonian Systems

The basic idea in the definition of incrementally port-Hamiltonian systems (32) is to replace the $\mathcal{N}$ given by Equation 38 with a maximal monotone relation $\mathcal{M}$ (see 33). Apart from the mathematical motivation, this idea is aimed at establishing a framework for studying incremental and contraction properties for control—e.g., for tracking purposes.

Definition 2. Let $\mathcal{F}$ be a finite-dimensional linear space. A relation $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$, with $\mathcal{E} = \mathcal{F}^*$, is monotone if[1]

$$(e_1 - e_2)^T (f_1 - f_2) \geq 0 \qquad 39.$$

for all $(f_i, e_i) \in \mathcal{M}$, $i = 1, 2$. This relation is called maximal monotone if it is monotone and the implication

$$\mathcal{M}' \text{ is monotone and } \mathcal{M} \subset \mathcal{M}' \;\Rightarrow\; \mathcal{M} = \mathcal{M}' \qquad 40.$$

holds. Consider a maximal monotone relation $\mathcal{M} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_P \times \mathcal{E}_P$ and a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, with $\mathcal{X} = \mathcal{F}_S$ a linear state space. The dynamics of the corresponding incrementally port-Hamiltonian system is defined as

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f_P(t), e_P(t)\right) \in \mathcal{M}, \quad t \in \mathbb{R}. \qquad 41.$$

Hence, incrementally port-Hamiltonian systems are characterized by the inequality

$$\left(\frac{\partial H}{\partial x}(x_1(t)) - \frac{\partial H}{\partial x}(x_2(t)\right)^T (\dot{x}_1(t) - \dot{x}_2(t)) \leq (e_P^1(t) - e_P^2(t))^T (f_P^1(t) - f_P^2(t)) \qquad 42.$$

along all pairs of trajectories $(x_i(t), f_P^i(t), e_P^i(t)), i = 1, 2$, of Equation 41. In particular, by taking one of the trajectories to be a steady-state solution $x(t) = \bar{x}$, $f_P(t) = \bar{f}_P$, $e_P(t) = \bar{e}_P$, we have

$$\left(\frac{\partial \widehat{H}_{\bar{x}}}{\partial x}(x(t))\right)^T \dot{x}(t) \leq (e_P(t) - \bar{e}_P)^T (f_P(t) - \bar{f}_P) \qquad 43.$$

for all solutions $x(t), f_P(t), e_P(t)$ of Equation 41, implying the property of shifted passivity.

What is the relation between port-Hamiltonian systems and incrementally port-Hamiltonian systems? First of all, if $\mathcal{D}$ is a constant Dirac structure on a linear state space and the port-Hamiltonian system has no energy dissipation, then the system is also incrementally port-Hamiltonian. This follows easily from the fact that any constant Dirac structure is a maximal monotone relation (but not the other way around).

In the case with energy dissipation, the relation between port-Hamiltonian systems and incrementally port-Hamiltonian systems is less simple, as can be seen from the following basic example.

Example 2 (circuit with tunnel diode). Consider an electrical LC circuit together with a resistor corresponding to an electrical port $f_R = -I, e_R = V$ (current and voltage, respectively). For a linear resistor (conductor) $I = gV$, $g > 0$, the system is both port-Hamiltonian and incrementally port-Hamiltonian. For a nonlinear conductor $I = G(V)$, the system is port-Hamiltonian if the graph of the function G is in the first and third quadrant and incrementally port-Hamiltonian if G is monotonically nondecreasing. For example, a tunnel diode characteristic

$$I = \Phi(V - V_0) + I_0,$$

for certain positive constants V_0, I_0, along with a function $\Phi(z) = \gamma z^3 - \alpha z, \alpha, \gamma > 0$, defines a system that is port-Hamiltonian but not incrementally port-Hamiltonian. Note that

[1]For simplicity, we use the notation $e^T f$ instead of $< e \mid f >$ throughout.

the addition of a constant current source I_0 and a constant voltage source V_0 to the circuit results in the classical construction of the van der Pol oscillator.

It should be remarked that physical systems with constant sources are, strictly speaking, not port-Hamiltonian but typically are incrementally port-Hamiltonian. Furthermore, Reference 34 discusses how systems with constant sources can be represented as the interconnection of a port-Hamiltonian system with a port-Hamiltonian source system (having a linear Hamiltonian that is not bounded from below and is therefore not a passive system).

An important subclass of incrementally port-Hamiltonian systems is given as

$$\begin{aligned} \dot{x} &= -\frac{\partial K}{\partial e}(e,u), \quad e = \nabla H(x), \\ y &= -\frac{\partial K}{\partial u}(e,u), \end{aligned} \tag{44.}$$

where $K(e,u)$ is a convex function of e, u. This follows from the fact that the differential of a convex function defines a maximally monotone relation (33). (This can be extended to nondifferentiable convex functions by replacing ordinary differentials with subdifferentials.) Especially amenable is the case that the convex function K is of the form $K(e,u) = P(e) - e^T Gu$, where P is a convex function of e, and G is an $n \times m$ matrix. This yields the incrementally port-Hamiltonian system

$$\begin{aligned} \dot{x} &= -\frac{\partial P}{\partial e}(e) + Gu, \quad e = \nabla H(x), \\ y &= G^T e. \end{aligned} \tag{45.}$$

This class of incrementally port-Hamiltonian systems offers an interesting stability analysis perspective. While typically the energy H is used as a candidate Lyapunov function, in this case P can also be used. Indeed, for $u = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\nabla H(x)) = -\frac{\partial P}{\partial e^T}(\nabla H(x))\frac{\partial^2 H}{\partial x^2}(x)\frac{\partial P}{\partial e}(\nabla H(x)) \leq 0 \tag{46.}$$

whenever the Hessian matrix $\frac{\partial^2 H}{\partial x^2}(x)$ is ≥ 0. In particular, equilibria $\bar{x}$ for $u = 0$ are such that $\frac{\partial P}{\partial e}(\nabla H(\bar{x})) = 0$, and whenever $P(\nabla H(x))$ has a minimum at $\bar{x}$, it serves as a Lyapunov function. Interestingly, as can also be seen from the basic example above, P is a Rayleigh [or (co-)content] function corresponding to energy dissipation, in line with stability theory in irreversible thermodynamics, where stability is characterized by minimal entropy production.

An important extension of Equation 45 is offered by the following class of incrementally port-Hamiltonian systems:

$$\begin{aligned} \dot{x} &= Je - \frac{\partial P}{\partial e}(e) + Gu, \quad e = \nabla H(x), \\ y &= G^T e, \end{aligned} \tag{47.}$$

where J is any constant skew-symmetric mapping, and P is a convex function, as above. In this case, the maximally monotone relation defined by P, G, and J cannot be integrated to a convex function K unless $J = 0$. Clearly, any incrementally port-Hamiltonian system given by Equation 47 is shifted Hamiltonian as well.

A nonphysical example of an incrementally port-Hamiltonian system of this type is provided by primal–dual gradient algorithms, as arising from a constrained optimization problem

$$\min_{q;\, Aq=b} C(q), \tag{48.}$$

where $C : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $Aq = b$ are affine constraints, for some $k \times n$ matrix A and vector $b \in \mathbb{R}^k$. The corresponding Lagrangian function is defined as

$$L(q, \lambda) := C(q) + \lambda^T (Aq - b), \quad \lambda \in \mathbb{R}^k, \tag{49.}$$

which is convex in q and concave in λ. The primal–dual gradient algorithm for solving the optimization problem in continuous time is given as

$$\begin{aligned} \tau_q \dot{q} &= -\frac{\partial L}{\partial q}(q, \lambda) = -\tfrac{\partial C}{\partial q}(q) - A^T \lambda + u, \\ \tau_\lambda \dot{\lambda} &= \frac{\partial L}{\partial \lambda}(q, \lambda) \quad = Aq - b, \\ y &= q, \end{aligned} \tag{50.}$$

where τ_q and τ_λ are positive diagonal matrices (determining the timescales of the algorithm). Here, we have added an input vector $u \in \mathbb{R}^n$ representing possible interaction with other algorithms or dynamics (e.g., if the primal–dual gradient algorithm is carried out in a distributed fashion). The output vector is defined as $y = q \in \mathbb{R}^n$.

Defining new state variables $x_q = \tau_q q, x_\lambda = \tau_\lambda \lambda$, one can rewrite Equation 50 as the incrementally port-Hamiltonian system in the form of Equation 47:

$$\begin{bmatrix} \dot{x}_q \\ \dot{x}_\lambda \end{bmatrix} = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_\lambda \end{bmatrix} - \begin{bmatrix} \frac{\partial P}{\partial e_q}(e_q, e_\lambda) \\ \frac{\partial P}{\partial e_\lambda}(e_q, e_\lambda) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} u, \quad y = e_q, \tag{51.}$$

with quadratic Hamiltonian

$$H(x_q, x_\lambda) := \frac{1}{2} x_q^T \tau_q^{-1} x_q + \frac{1}{2} x_\lambda^T \tau_\lambda^{-1} x_\lambda, \quad e_q = \frac{\partial H}{\partial x_q}(x_q, x_\lambda) = q, \; e_\lambda = \frac{\partial H}{\partial x_\lambda}(x_q, x_\lambda) = \lambda \tag{52.}$$

and convex function $P(e_q, e_\lambda) := C(e_q) + b^T e_\lambda$.

Example 3. Consider the swing equation model of the power network as given in Example 1. Let $u = u_g - u_d$, where u_g is the vector of generated power and u_d the vector of consumed power at the nodes. Define the social welfare $U(u_d) - C(u_g)$, consisting of the utility function $U(u_d)$ of the consumers u_d, and the power generation cost $C(u_g)$ associated with the producers u_g. Assume that $C(u_g)$ is strictly convex and $U(u_d)$ is strictly concave. Consider the objective of maximizing the social welfare under the constraint of zero frequency deviation. As mentioned above, a necessary and sufficient condition for zero frequency deviation $\omega_* = 0$ is $\mathbb{1}^T u_d = \mathbb{1}^T u_g$. Furthermore, $\mathbb{1}^T u_d = \mathbb{1}^T u_g$ if and only if there exists a v such that $u_g - u_d = D_c v$, where D_c is the incidence matrix of any connected graph with the same n nodes as the power network. This leads to the problem of minimizing $C(u_g) - U(u_d)$ over all u_g, u_d, v, such that $D_c v - u_g + u_d = 0$. The primal–dual gradient algorithm for this convex minimization problem is

$$\begin{aligned} \tau_g \dot{u}_g &= -\nabla C(u_g) + \lambda + w_g, \\ \tau_d \dot{u}_d &= \nabla U(u_d) - \lambda + w_d, \\ \tau_v \dot{v} &= -D_c^T \lambda, \\ \tau_\lambda \dot{\lambda} &= D_c v - u_g + u_d, \end{aligned} \tag{53.}$$

with additional inputs $w = (w_g, w_d)$ and matrices $\tau_g, \tau_d, \tau_v, \tau_\lambda > 0$ corresponding to the timescales of the algorithm. This defines a dynamic pricing controller, where λ_i acts as the price at the ith node, and v represents the information exchange of the differences of the prices λ along the edges of the graph specified by D_c.

This controller constitutes an incrementally port-Hamiltonian system. This fact was used in Reference 20 for the control of a power network by interconnecting the controller given by Equation 53 to the power network and using the fact that they are both shifted port-Hamiltonian. Convergence to the optimal values of u_g and u_d is then guaranteed by using the sum of the shifted Hamiltonians of the physical power network and of the dynamic pricing controller as a Lyapunov function. Note that the closed-loop system constitutes a true cyber-physical system: the physical power network together with the market dynamics represented by the dynamic pricing controller, both represented in (shifted) port-Hamiltonian form.

4.2. Port-Hamiltonian Systems as Pseudo-Gradient Systems

A pseudo-gradient system with inputs and outputs is given as (35–37)

$$\begin{aligned} G(z)\dot{z} &= -\frac{\partial V}{\partial z}(z,u), \\ y &= -\frac{\partial V}{\partial u}(z,u), \end{aligned} \qquad 54.$$

for some potential function $V(z,u)$. Here $G(z) = G^T(z)$ defines a pseudo-Riemannian metric on $\mathcal{X}$ and a true Riemannian metric whenever $G(z) > 0$. The pseudo-Riemannian metric is called Hessian if there exists a function $K(z)$ such that the (i,j)th element $g_{ij}(z)$ of the matrix $G(z)$ is given as

$$g_{ij}(z) = \frac{\partial^2 K}{\partial z_i \partial z_j}(z), \quad i,j = 1,\ldots,n. \qquad 55.$$

A necessary and sufficient condition for the local existence of such a function $K(z)$ is the integrability condition (38)

$$\frac{\partial g_{jk}}{\partial z_i}(z) = \frac{\partial g_{ik}}{\partial z_j}(z), \quad i,j,k = 1,\ldots,n. \qquad 56.$$

Indeed, Equation 56 guarantees the local existence of functions $g_k(z)$ such that $g_{jk}(z) = \frac{\partial g_k}{\partial z_j}(z)$, $j,k = 1,\ldots,n$. Then, by symmetry of $G(z)$,

$$\frac{\partial g_k}{\partial z_j}(z) = g_{jk}(z) = g_{kj}(z) = \frac{\partial g_j}{\partial z_k}(z), \quad j,k = 1,\ldots,n, \qquad 57.$$

which is the integrability condition guaranteeing the local existence of a function $K(z)$ satisfying $g_j(z) = \frac{\partial K}{\partial z_j}(z)$, $j = 1,\ldots,n$, which, by differentiation with respect to z_i and in view of the definition of $g_j(z)$, $j = 1,\ldots,n$, amounts to Equation 55.

Example 4. The restriction to Hessian pseudo-Riemannian metrics is exemplified by the Brayton–Moser formulation of RLC circuits (39, 40) given as the pseudo-gradient system (here without inputs and outputs)

$$\begin{bmatrix} L & 0 \\ 0 & -C \end{bmatrix} \dot{z} = -\frac{\partial P}{\partial z}(z), \quad z = \begin{bmatrix} I \\ V \end{bmatrix}, \qquad 58.$$

with $P(z)$ the mixed-potential function, defined as

$$P(z) = P_1(I) + P_2(V) + I^T \Lambda V \qquad 59.$$

for a certain matrix Λ reflecting the topology of the circuit, a resistive content function P_1, and a conductive co-content function $-P_2$ (corresponding to the nonlinear resistors).

Now let us show how, under additional conditions, a port-Hamiltonian system can be represented as a pseudo-gradient system with Hessian pseudo-Riemannian metrics. We consider port-Hamiltonian systems as in Equation 17 but allow for nonlinear energy dissipation

$$\begin{aligned} \dot{x} &= J(x)\nabla H(x) - R(\nabla H(x)) + B(x)u, \\ y &= B^T(x)\nabla H(x), \end{aligned} \qquad 60.$$

where the mapping R satisfies $e^T R(e) \geq 0$ for all e, and, in order to avoid confusion, the input matrix is denoted by $B(x)$.

First, we need to assume that the mapping from x to $e := \frac{\partial H}{\partial x}(x)$ is invertible. Then its inverse is given by

$$x = \frac{\partial H^*}{\partial e}(e), \qquad 61.$$

where $H^*(e) = e^T x - H(x)$ is the Legendre transform of H. From substituting $x(t) = \frac{\partial H^*}{\partial e}(e(t))$ into Equation 60, it follows that

$$\frac{\partial^2 H^*}{\partial e^2}(e)\dot{e} = J(x)e - R(e) + B(x)u. \qquad 62.$$

Substituting $x = \frac{\partial H^*}{\partial e}(e)$, one obtains a differential equation in the new state variables e.

Second, assume that there exist coordinates $x = (x_q, x_p)$ in which the matrices $J(x)$ and $B(x)$ are constant and take the form

$$J = \begin{bmatrix} 0 & -P_c \\ P_c^T & 0 \end{bmatrix}, \; B = \begin{bmatrix} B_q \\ 0 \end{bmatrix}. \qquad 63.$$

Third, assume that the Hamiltonian H splits as $H(x_q, x_p) = H_q(x_q) + H_p(x_p)$, for certain functions H_q and H_p. Writing accordingly $e = (e_q, e_p)$, with $e_q = \frac{\partial H_q}{\partial x_q}(x_q)$, $e_p = \frac{\partial H_p}{\partial x_p}(x_p)$, it follows that the Legendre transform $H^*(e)$ splits as $H^*(e) = H_q^*(e_q) + H_p^*(e_p)$. Then Equation 62 takes the form

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} 0 & -P_c \\ P_c^T & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} - \begin{bmatrix} R_q(e) \\ R_p(e) \end{bmatrix} + \begin{bmatrix} B_q \\ 0 \end{bmatrix} u. \qquad 64.$$

Finally, assume that

$$R_q(e) = \frac{\partial P_q}{\partial e_q}(e_q), \; R_p(e) = -\frac{\partial P_p}{\partial e_p}(e_p) \qquad 65.$$

for certain (Rayleigh dissipation) functions P_q, P_p. Then, from defining the mixed-potential function as

$$P(e_q, e_p) := P_q(e_q) + P_p(e_p) + e_q^T P_c e_p, \qquad 66.$$

it follows, after multiplication of the last line in Equation 64 by -1, that Equation 64 can be rewritten as

$$\begin{bmatrix} \frac{\partial^2 H_q^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H_p^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}e_q}{\mathrm{d}t} \\ \frac{\mathrm{d}e_p}{\mathrm{d}t} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial e_q} \\ \frac{\partial P}{\partial e_p} \end{bmatrix} + \begin{bmatrix} B_q \\ 0 \end{bmatrix} u, \qquad 67.$$
$$y = B_q^T e_q.$$

These equations define a pseudo-gradient system with respect to a Hessian pseudo-Riemannian metric and obviously generalize the Brayton–Moser equations shown in Equation 58.

Example 5. Consider the swing equation model of a power network as discussed in Example 1:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \Gamma \operatorname{Sin} q \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \quad y = \omega, \qquad 68.$$

where D is the $n \times m$ incidence matrix of the power network with n nodes and m edges (transmission lines). This is an incrementally and ordinary port-Hamiltonian system with Hamiltonian

$$H(q,p) = H_q(q) + H_p(p) := -\sum_i \gamma_i \cos q_i + \frac{1}{2} p^T M^{-1} p \qquad 69.$$

and Rayleigh dissipation function $P_p(\omega) = \frac{1}{2}\omega^T A \omega$. The e variables are given by

$$\begin{aligned} \pi &:= \frac{\partial H_q}{\partial q}(q) = \Gamma \operatorname{Sin} q \quad \text{(power flows through the lines)}, \\ \omega &:= \frac{\partial H_p}{\partial p}(p) = M^{-1} p \quad \text{(frequency deviations at the nodes)}. \end{aligned} \qquad 70.$$

Then Equation 68 can be rewritten as the pseudo-gradient system

$$\begin{bmatrix} -K(\pi) & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{\pi} \\ \dot{\omega} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P}{\partial \pi}(\pi,\omega) \\ \frac{\partial P}{\partial \omega}(\pi,\omega) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \quad y = \omega, \qquad 71.$$

where $K(\pi)$ is the positive diagonal matrix with kth diagonal element $\frac{1}{\sqrt{\gamma_k^2 - \pi_k^2}}$, with mixed-potential function

$$P(\pi,\omega) = \pi^T D \omega + \frac{1}{2}\omega^T A \omega. \qquad 72.$$

Note that the primal–dual gradient algorithm as discussed in the previous section is (by definition) a pseudo-gradient system as well, and its discussed transformation into (incrementally) port-Hamiltonian form exactly follows the same steps as above (but in opposite order).

4.3. Implicit Energy-Storage Relations

A further extension of port-Hamiltonian systems is obtained by generalizing the standard energy-storage relation $\dot{x} = -f_S, e_S = \nabla H(x), x \in \mathcal{X}$, where $\nabla H(x)$ is the column vector of partial derivatives of a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, by an implicit relation

$$\dot{x} = -f_S, \quad (x, e_S) \in \mathcal{L}, \qquad 73.$$

where $\mathcal{L}$ is a Lagrangian submanifold of the cotangent bundle $T^*\mathcal{X}$ of the state-space manifold $\mathcal{X}$. Noting that for any H the submanifold $\{(x, \nabla H(x)) \mid x \in \mathcal{X}\}$ is Lagrangian shows that this is indeed a direct generalization of the standard energy-storage relation. In fact, an arbitrary Lagrangian submanifold $\mathcal{L} \subset T^*\mathcal{X}$ is of this form for a certain H if and only if the image of $\mathcal{L}$ under the canonical projection $T^*\mathcal{X} \to \mathcal{X}$ is equal to $\mathcal{X}$. Implicit energy-storage relations naturally appear in a number of cases, both in physical systems (41) and elsewhere.

Example 6 (optimal control, from References 41 and 42). Consider the optimal control problem of minimizing a cost functional $\int L(q, u)\mathrm{d}t$ for the control system $\dot{q} = f(q, u)$, with $q \in \mathbb{R}^n, u \in \mathbb{R}^m$. Define the optimal control Hamiltonian

$$K(q, p, u) = p^T f(q, u) + L(q, u), \tag{74.}$$

where $p \in \mathbb{R}^n$ is the co-state vector. Application of Pontryagin's maximum principle leads to the consideration of the port-Hamiltonian system (without inputs and outputs) on the space (q, p, u), given as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p, u) \\ \frac{\partial H}{\partial p}(q, p, u) \\ \frac{\partial H}{\partial u}(q, p, u) \end{bmatrix}. \tag{75.}$$

This can be equivalently rewritten as a port-Hamiltonian system involving only the (q, p) variables, with implicit energy-storage relations given by the Lagrangian submanifold

$$\mathcal{L} = \left\{ \left(\begin{bmatrix} q \\ p \end{bmatrix}, \begin{bmatrix} e_q \\ e_p \end{bmatrix} \right) \middle| \exists u \text{ s.t. } \begin{bmatrix} e_q \\ e_p \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial q}(q, p, u) \\ \frac{\partial H}{\partial p}(q, p, u) \end{bmatrix}, \ \frac{\partial H}{\partial u}(q, p, u) = 0 \right\}. \tag{76.}$$

Port-Hamiltonian systems with implicit energy-storage relations $\mathcal{L} \subset T^*\mathcal{X}$ as above can be locally represented as follows. Any Lagrangian submanifold $\mathcal{L} \subset T^*\mathcal{X}$, with $\dim \mathcal{X} = n$, can be locally written as (16)

$$\mathcal{L} = \left\{ (x, e) = (x_I, x_J, e_I, e_J) \in T^*\mathcal{X} \middle| e_I = \frac{\partial V}{\partial x_I}, x_J = -\frac{\partial V}{\partial e_J} \right\}, \tag{77.}$$

for some splitting $\{1, \ldots, n\} = I \cup J$ of the index set (possibly after reordering) and a function $V(x_I, e_J)$, called the generating function. In particular, x_I, e_J serve as local coordinates for $\mathcal{L}$. Now define the Hamiltonian $\widetilde{H}(x_I, e_J)$ as

$$\widetilde{H}(x_I, e_J) := V(x_I, e_J) - e_J^T \frac{\partial V}{\partial e_J}(x_I, e_J). \tag{78.}$$

Consider any modulated Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{E}_R \times \mathcal{F}_P \times \mathcal{E}_P$. Since the dynamics of the port-Hamiltonian system with implicit energy-storage relation $\mathcal{L}$ satisfies the property $-e_S^T f_S = e_R^T f_R + e_P^T f_P$, and by Equation 77 the coordinate expressions of f_S, e_S (in terms of x_I, e_J) are given as

$$-f_S = \begin{bmatrix} I & 0 \\ -\frac{\partial^2 V}{\partial e_J \partial x_I} & -\frac{\partial^2 V}{\partial e_J^2} \end{bmatrix} \begin{bmatrix} \dot{x}_I \\ \dot{e}_J \end{bmatrix}, \quad e_S = \begin{bmatrix} \frac{\partial V}{\partial x_I} \\ e_J \end{bmatrix}, \tag{79.}$$

it can be verified that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{H}(x_I, e_J) = e_R^T(t) f_R(t) + e_P^T(t) f_P(t) \leq e_P^T(t) f_P(t). \qquad 80.$$

Hence $\widetilde{H}(x_I, e_J)$ serves as the energy expression[2] of the system in local coordinates x_I, e_J. This observation generalizes the results for linear port-Hamiltonian systems with implicit energy storage, as obtained in References 41 and 43 for the nonlinear case.

From a port-based modeling perspective, the algebraic constraints in a port-Hamiltonian system model arise primarily from the properties of the Dirac structure. Specifically, let us concentrate on the case of only energy-storing elements with Hamiltonian H and no energy dissipation or external ports. Then the algebraic constraints, called Dirac algebraic constraints in Reference 41, are specified as

$$\nabla H(x) \in \pi^*(\mathcal{D}(x)), \qquad 81.$$

where $\pi^*(\mathcal{D}(x))$ is the projection of the Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$ at $x \in \mathcal{X}$ on the cotangent space $T_x^*\mathcal{X}$. Clearly, if $\pi^*(\mathcal{D}(x)) = T_x^*\mathcal{X}$, then there are no algebraic constraints, and in fact the Dirac structure can be written as the graph of a skew-symmetric map $J(x)$ from $T_x^*\mathcal{X}$ to $T_x\mathcal{X}$.

On the other hand, if H is replaced by a Lagrangian submanifold $\mathcal{L}$, then another type of algebraic constraints, called Lagrange algebraic constraints in Reference 41, may appear. These algebraic constraints are simply given by

$$x \in \pi(\mathcal{L}), \qquad 82.$$

where $\pi : T^*\mathcal{X} \to \mathcal{X}$ is the canonical projection. Thus, Lagrange algebraic constraints arise whenever the Lagrangian submanifold $\mathcal{L}$ is such that $\pi(\mathcal{L}) \neq \mathcal{X}$, or, equivalently, whenever $\mathcal{L}$ cannot be written as the graph of ∇H for some $H : \mathcal{X} \to \mathbb{R}$.

Let us now show how one converts Dirac algebraic constraints (as favored by port-based modeling) into Lagrange algebraic constraints (which have nicer properties from the point of view of numerical simulation) by adding extra state variables. This extends the construction detailed in Reference 41 from the linear to the nonlinear case. The first observation (7) is that a general Dirac structure $\mathcal{D}$ can be written as the graph of a skew-symmetric map on an extended state space as follows. Suppose $\pi^*(\mathcal{D}(x)) \subset T_x\mathcal{X}^*$ is $(n-k)$-dimensional. Define $\Lambda := \mathbb{R}^k$. Then there exists a full-rank $n \times k$ matrix $B(x)$ and a skew-symmetric $n \times n$ matrix $J(x)$ such that

$$\mathcal{D}(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid \exists \lambda^* \in \Lambda^* \text{ s.t. } -f = J(x)e + B(x)\lambda^*,\ 0 = B^T(x)e\}. \qquad 83.$$

Conversely, any such equations for a skew-symmetric map $J(x) : T_x^*\mathcal{X} \to T_x\mathcal{X}$ define a Dirac structure. Now, let the energy-storage relation of the port-Hamiltonian system be given in an ordinary way, i.e., by a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$. Then, with respect to the extended state space $\mathcal{X}_e := \mathcal{X} \times \Lambda$, we may define the implicit energy-storage relation given by the Lagrangian submanifold (of the same type as in Equation 77)

$$\mathcal{L}_e := \{(x, \lambda, e, \lambda^*) \in T^*\mathcal{X}_e \mid e = \nabla H(x), \lambda = 0\}, \qquad 84.$$

corresponding to the Lagrange algebraic constraint $0 = \lambda$ $[= B^T(x)\nabla H(x)]$. Hence, the Dirac algebraic constraint $0 = B^T(x)\nabla H(x)$ has been transformed into the Lagrange algebraic constraint

[2]Note that if the relation $x_J = -\frac{\partial V}{\partial e_J}$ is invertible, and hence the Lagrangian submanifold is parameterized by $x = (x_I, x_J)$ and thus is of the form $\mathcal{L} = \{(x, \nabla H(x)) \mid x \in \mathcal{X}\}$ for a certain H, then actually $\widetilde{H}(x_I, x_J)$ is the partial Legendre transform of $V(x_I, e_J)$ with respect to e_J and equals H.

$\lambda = 0$ on the extended state space $\mathcal{X}_e$. The generating function of $\mathcal{L}_e$ is H, which is independent of λ^*, and therefore $\widetilde{H}(x, \lambda^*) := H(x) - \lambda^{*T} \frac{\partial H}{\partial \lambda^*}(x) = H(x)$.

Note that the above transformation of Dirac algebraic constraints into Lagrange algebraic constraints by extension of the state space is opposite to the situation considered in Example 6, where Dirac algebraic constraints were transformed into Lagrange algebraic constraints by reduction (leaving out the u variables), or, said differently, where Lagrange algebraic constraints were transformed into Dirac algebraic constraints by extension of the state space. (For more information, see Reference 41.)

4.4. Port-Hamiltonian Systems and Thermodynamics

Port-Hamiltonian systems theory has been successfully applied to a wide range of complex multiphysics systems. Nevertheless, the application to thermodynamic systems poses fundamental questions. This can be illustrated by the basic example of two heat compartments with a conducting wall (44). The two systems, indexed by 1 and 2, exchange heat flow q given by Fourier's law $q = \lambda(T_1 - T_2)$, where the temperatures are given as $T_i = \frac{\partial U_i}{\partial S_i}(S_i), i = 1, 2$, with $U_1(S_1), U_2(S_2)$ the internal energies of the two compartments. This leads to the system

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \begin{bmatrix} -\frac{q}{T_1} \\ \frac{q}{T_2} \end{bmatrix} = \begin{bmatrix} -\lambda \frac{T_1 - T_2}{T_1} \\ \lambda \frac{T_1 - T_2}{T_2} \end{bmatrix} = \begin{bmatrix} 0 & \lambda(\frac{1}{T_1} - \frac{1}{T_2}) \\ -\lambda(\frac{1}{T_1} - \frac{1}{T_2}) & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial U}{\partial S_1} \\ \frac{\partial U}{\partial S_2} \end{bmatrix}, \qquad 85.$$

with total energy $U(S_1, S_2) := U_1(S_1) + U_1(S_2)$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t} U = 0$. This is, however, not a true port-Hamiltonian system, since the skew-symmetric map

$$\begin{bmatrix} 0 & \lambda(\frac{1}{T_1} - \frac{1}{T_2}) \\ -\lambda(\frac{1}{T_1} - \frac{1}{T_2}) & 0 \end{bmatrix} \qquad 86.$$

does not depend on S_1, S_2 directly, but instead does so through $T_i = \frac{\partial U_i}{\partial S_i}(S_i)$. Therefore, it does not define a true Poisson or Dirac structure on the state space $\mathbb{R}^2$ with coordinates S_1, S_2. Instead, Equation 85 is an example of the type

$$\dot{x} = J(e)e, \quad J(e) = -J^T(e), \quad e = \nabla H(x), \qquad 87.$$

where the right-hand side $J(e)e$ depends nonlinearly on e. Furthermore, this nonlinear dependence on e appears to be crucial. In fact, in the example of the two heat compartments, it is responsible for the fundamental property

$$\dot{S}_1 + \dot{S}_2 = \frac{(T_1 - T_2)^2}{T_1 T_2} \geq 0, \qquad 88.$$

expressing the irreversible increase of the total entropy. This has recently led (see 45) to a generalization of the port-Hamiltonian framework that makes use of contact and homogeneous symplectic geometry as well as implicit energy-storage relations (see also 44).

More generally, the framework of port-based modeling and port-Hamiltonian systems as reviewed in this article emphasizes the first law of thermodynamics, i.e., the interconnection of systems by energy flow and the conservation of total energy. The second law of thermodynamics is reflected only by the presence of energy-dissipating elements, which irreversibly transform part of the energy into heat. Furthermore, control by interconnection of port-Hamiltonian systems considers controller systems primarily as energy-processing port-Hamiltonian systems, where the controller system may be an actual physical system or a cyber system that is emulating

the behavior of a physical port-Hamiltonian system. An example of the latter is the classical interpretation of a proportional–integral controller as mimicking a damper and a spring. The clear advantages of this paradigm of control by interconnection with port-Hamiltonian controller systems are the inherent robustness when interacting with an unknown but passive environment and the robustness with regard to physical parameter variations (in either the plant or controller port-Hamiltonian system). In contrast to control by interconnection, energy-routing control aims at directly influencing the energy flow in the plant port-Hamiltonian systems. In this case, another aspect is already coming into play: The energy-routing control is based on information about the system (as well as feedforward information regarding the control task). This relates to another prevailing paradigm of control as information gathering and processing. The open problem is how to unite these two dominant control paradigms. It is tempting to assume that the extension of port-Hamiltonian systems theory to thermodynamics may provide the key to solving this problem.

SUMMARY POINTS

1. Port-Hamiltonian systems theory provides a systematic framework for modeling and analysis of (possibly large-scale) multiphysics systems.
2. The geometric theory of port-Hamiltonian systems extends the classical theory of Hamiltonian dynamics through the inclusion of energy dissipation, external ports, and algebraic constraints on the state variables.
3. The underlying geometric structure is that of a Dirac structure, generalizing symplectic and Poisson structures and incorporating network topology.
4. The basic properties of port-Hamiltonian systems include passivity, shifted passivity, the existence of Casimirs, and compositionality.
5. These properties can be fruitfully used for control, yielding robust and physically interpretable control strategies and viewing controller systems primarily as emulating additional physical dynamics.
6. The port-Hamiltonian framework suggests new control paradigms such as impedance and energy-routing control.
7. Under additional conditions, port-Hamiltonian systems can be represented as pseudo-gradient systems, which also relate to optimization algorithms.

FUTURE ISSUES

1. The potential of incrementally port-Hamiltonian systems, together with the related notion of differential passivity (46), for purposes of tracking control should be further explored.
2. Consideration of implicit energy-storage relations raises new issues within port-Hamiltonian dynamics analysis and simulation.
3. The inclusion of thermodynamics in the port-Hamiltonian framework calls for new geometric formulations.
4. This inclusion of thermodynamics may also unite the paradigm of controllers as energy-processing components with that of controllers as processors of information.

DISCLOSURE STATEMENT

The author is not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

ACKNOWLEDGMENTS

This article owes much to collaborations with several people over many years. In particular, I would like to thank Bernhard Maschke (Université de Lyon 1) for a long-lasting and inspiring collaboration on many of the topics discussed in this article.

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