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The Econometrics of Static Games

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Abstract

This article reviews the econometrics of static games, with a focus on discrete-choice cases. These models have been used to study a rich variety of empirical problems, ranging from labor force participation to entry decisions. We outline the components of a general game and describe the problem of doing robust inference in the presence of multiple solutions, as well as the different econometric approaches that have been applied to tackle this problem. We then describe the specific challenges that arise in different variations of these models depending on whether players are assumed to have complete or incomplete information, as well as whether or not nonequilibrium play is allowed. We describe the results in 2×2 games (the most widely studied games in econometrics), and we present extensions and recent results in games with richer action spaces. Areas for future research are also discussed.

1. INTRODUCTION

This article provides an overview of the general problem and the state of the literature on econometric inference in static games, with an emphasis on discrete-choice games. These models have been applied to study problems including labor force participation, entry, technology adoption, product differentiation, advertising, and analyst stock recommendations, among others. In these models, a data set with a sample of outcomes (i.e., choices made by economic agents) and observable covariates relevant to the environment is observed and mapped back to an underlying model of strategic interaction (a game) through behavioral assumptions. Typically, in the econometric analysis of static games, each observation in the sample is considered an independent and identically distributed (i.i.d.) realization from the same underlying game, summarized in a normal-form representation. A characterization of the normal form requires a description of the collection of players, their action space, and their payoff functions. These are the primary components of the model. Here, the researcher must determine, for example, whether the model is a binary-choice game and whether payoff functions will be parametrically or nonparametrically specified. Once the normal form is specified, the researcher must make assumptions regarding the information possessed by the players, the solution concept used, and, if multiple solutions exist, the type of assumptions made about the underlying selection mechanism. While existing studies differ in terms of these assumptions, the ultimate goal in all cases is to conduct inference on features of the underlying game, and particularly on the nature of the strategic interaction effects. Depending on the extent to which the underlying game has been specified, inference can also lead to counterfactual analysis, or it can enable the researcher to test whether the data observed are consistent with a specific behavioral model such as Nash equilibrium.

Aside from being valuable tools that can enable researchers to estimate strategic interaction effects, test and compare alternative behavioral models, and perform policy counterfactuals, these models pose very particular econometric challenges that we aim at describing here, along with the solutions that have been proposed to tackle them. In particular, we compare and contrast existing econometric methods that have been developed to deal with the following issues:

- Nash equilibrium versus weaker solution concepts;
- the presence of multiple solutions;
- complete- versus incomplete-information games;
- correct versus incorrect beliefs; and
- parametric versus nonparametric models.

The article proceeds as follows. We begin in Section 2 by describing the generic econometric problem of doing inference in a static, normal-form game. We enumerate the structural and behavioral elements of the model and discuss how existing papers have differed in their assumptions. From here, Section 3 describes identification and the general inferential problem in these models, along with the challenges faced by the econometrician—in particular, the issue of robust inference when the game has multiple solutions. We also compare the particular challenges of complete-information and incomplete-information games, as well as equilibrium and nonequilibrium models. Section 4 then illustrates inference in all these cases in a 2×2 game, the canonical application in econometric models of games. We then proceed in Section 5 to describe recent advances in discrete games with richer action spaces. Overall, the results we overview throughout the article include parametric and nonparametric models, complete- and incomplete-information games, and equilibrium and nonequilibrium behavior. Following our review and analysis, we conclude in Section 6 by suggesting areas for future research.

2. A GENERAL NORMAL-FORM GAME

This article focuses on econometric inference of a normal-form (or strategic-form) static game where players move simultaneously (i.e., before observing the realized choices of others). The generic game consists of the following components.

Definition 1 (players and actions). The game consists of a set of $\{1, 2, \ldots, P\}$ players. Each player p has to select an action Y_p from within an action space $\mathcal{S}_p$. The subscript $_{-p}$ will denote p's opponents. We denote a generic action for p as $y_p \in \mathcal{S}_p$, and $y_{-p} \equiv (y_q)_{q \neq p} \in \mathcal{S}_{-p}$ denotes a generic action profile for p's opponents. Y_p will denote the actual action (the choice) made by p and $Y_{-p} \equiv (y_q)_{q \neq p}$ will denote the choice profile of p's opponents. We refer to $Y \equiv (Y_p)_{p=1}^{P}$ as the outcome of the game.

Definition 2 (payoff functions). The von Neumann–Morgenstern payoff function of player p is denoted as

$$u_p(Y_p, Y_{-p}). \tag{1}$$

Payoffs are treated as random functions, with $u_p \in \mathscr{U}_p$ (a space of functions). We assume that player p observes the realization of their payoff function u_p prior to the game being played. Let $\mathscr{U}$ denote the joint space of payoff functions for all the players in the game.

We consider static games where players choose their strategies simultaneously, and we focus on cases where actions are scalar and the action space is discrete. The game has the following additional components.

Definition 3 (beliefs). Players are expected-payoff maximizers. They construct their expected payoffs by forming beliefs about the distribution of actions chosen by their opponents. Beliefs for player p are denoted as π_{-p}, a distribution function over $\mathcal{S}_{-p}$. For a given set of beliefs π_{-p}, the expected payoff function for p is given by

$$\overline{u}_p(Y_p, \pi_{-p}) = \int_{y_{-p} \in \mathcal{S}_{-p}} u_p(Y_p, y_{-p}) d\pi_{-p}(y_{-p}). \tag{2}$$

We denote the space of possible beliefs for player p as $\mathcal{B}_{-p}$, with $\mathcal{B}$ denoting the combined space of beliefs for all players in the game. Degenerate beliefs that assign probability mass 1 to a particular action profile are always allowed.

Definition 4 (strategies). A strategy σ_p for player p is a distribution over $\mathcal{S}_p$. Let Σ_p denote the strategy space for player p, and let Σ denote the combined strategy space for all players in the game. For a given set of beliefs π_{-p}, the expected payoff for p from playing strategy σ_p is

$$\overline{u}_p(\sigma_p, \pi_{-p}) = \int_{y_p \in \mathcal{S}_p} \overline{u}_p(y_p, \pi_{-p}) d\sigma_p(y_p). \tag{3}$$

Pure strategies assign probability mass 1 to a particular action. We refer to all others as mixed strategies.

Definition 5 (best responses). The strategy σ_p is a best response for a particular set of beliefs π_{-p} if $\overline{u}_p(\sigma_p, \pi_{-p}) \geq \overline{u}_p(\sigma'_p, \pi_{-p})$ for all $\sigma'_p \in \mathcal{S}_p$. Given a payoff function u_p, we denote this as $\sigma_p = BR_p(\pi_{-p}|u_p)$.

Definition 6 (solution). Let Ω_p denote the joint space of beliefs and strategies (π_{-p}, σ_p) for player p. Given a payoff function u_p, we obtain

$$\Omega_p^*(u_p) = \left\{(\pi_{-p}, \sigma_p) \in \Omega_p \colon \sigma_p = BR_p(\pi_{-p}|u_p)\right\}.$$

Given $\{u_p\}_{p=1}^P$, a solution to the game is a collection of beliefs and strategies $\{(\pi_{-p}, \sigma_p)\}_{p=1}^P$ such that $(\pi_{-p}, \sigma_p) \in \Omega_p^*(u_p)$ for each p. Denote $u \equiv \{u_p\}_{p=1}^P$ and let $\Omega^*(u)$ be the set of all solutions to the game given u.

For a given strategy profile $y = (y_p)_{p=1}^P$, we have

$$\Omega^*(y|u) = \left\{\{(\pi_{-p}, \sigma_p)\}_{p=1}^P \in \Omega^*(u) : \sigma_p(y_p) > 0 \;\; \forall\, p\right\}. \qquad 4.$$

$\Omega^*(y|u)$ is the set of all existing solutions where y can be chosen with nonzero probability. Given u, we have $Y = y$ only if $\Omega^*(y|u) \neq \emptyset$. If the game has multiple existing solutions, the model can be completed by adding a solution selection mechanism.

Definition 7 (solution selection). Given u, there exists a mechanism $\mathcal{M}$ that selects a solution from within $\Omega^*(u)$. The solution selected by $\mathcal{M}$ is defined as

$$\{(\pi_{-p}^*(\cdot|u), \sigma_p^*(\cdot|u))\}_{p=1}^P.$$

Then, we obtain

$$Pr(Y_p \in A|u) = \int_{y_p \in A} d\sigma_p^*(y_p|u) \quad \forall\, A \subseteq \mathcal{S}_p. \qquad 5.$$

All the models we review here can be expressed as variations of the general normal-form game described above, with the following different assumptions.

Assumption 1 (solution concept). The majority of econometric studies assume equilibrium behavior, which presupposes correct beliefs, and the most widely assumed equilibrium concept is Nash equilibrium (NE). Nonequilibrium solution concepts have also been considered. For example, motivated by empirical evidence of deviations from equilibrium behavior (see, e.g., Stahl & Wilson 1994), experimental economists have used models of cognitive hierarchy (among others) to model behavior. These models are characterized by very precise assumptions about how exactly agents deviate from equilibrium. Examples include work by Costa-Gomes & Crawford (2001), Camerer (2004), and Costa-Gomes & Crawford (2006).

Econometric methods that do not impose equilibrium behavior include those developed by Aradillas-López & Tamer (2008), Kline & Tamer (2012), Kline (2015, 2018), and Aradillas-López (2019). Unlike the experimental methods, which assume a precise form in which players deviate from equilibrium, these papers rely on general restrictions on behavior and beliefs that include NE as a special case and, in some instances (e.g., Aradillas-López & Tamer 2008 and Kline 2015), also nest experimental models such as cognitive hierarchy.

Assumption 2 (information). Complete-information models assume that the realization of payoff functions $\{u_p\}$ is common knowledge, while incomplete-information settings assume that the exact realization of payoff functions is only privately observed by each player. Some examples of complete-information games include work by Bjorn & Vuong (1984), Bresnahan & Reiss (1990; 1991a,b), Berry (1992), Tamer (2003), Ciliberto & Tamer (2009), Bajari et al. (2010b), Aradillas-López (2011, 2019), Kline (2015, 2016), and Aradillas-López & Rosen (2019). Incomplete-information games have been studied, for example, by

Aradillas-López (2010, 2012), Bajari et al. (2010a), de Paula & Tang (2012), Xu (2014), Xu & Wan (2014), Lewbel & Tang (2015), Aradillas-López & Gandhi (2016), Liu et al. (2017), and Xiao (2018). The presence of multiple equilibria tends to be more pervasive in games with complete information (see Morris & Shin 2003). Incomplete-information games require assumptions about the information possessed by players to construct their beliefs.[1]

Assumption 3 (beliefs). Equilibrium models presuppose correct beliefs. In equilibrium games of complete information, beliefs are completely characterized by the realization of payoff functions. Incomplete-information games require specific assumptions about the information observed by players. A crucial assumption is whether players' private information is assumed to be independent across players, conditional on observables to the econometrician. Examples that assume conditional independence include work by Bajari et al. (2010a), Aradillas-López (2012), de Paula & Tang (2012), Lewbel & Tang (2015), Aradillas-López & Gandhi (2016), and Xiao (2018). Papers that allow correlation in players' private information under additional restrictions include those by Aradillas-López (2012), Grieco (2014), Xu (2014), Xu & Wan (2014), and Liu et al. (2017). Methods that allow for incorrect beliefs typically restrict the space of beliefs according to weaker restrictions, such as iterated dominance or rationalizability.

Assumption 4 (action spaces). We focus mainly on discrete action spaces. Within these models, binary-choice games have received the majority of attention (see, e.g., Berry & Tamer 2007 and the literature cited therein). Discrete games with a richer-than-binary action space typically have a more complicated set of solutions and therefore are more challenging to analyze. Complete-information games with more than two actions have been analyzed by Davis (2006) and Bajari et al. (2010b) (multinomial games), and by Aradillas-López (2011) and Aradillas-López & Rosen (2019) (ordinal games). Nonparametric results in ordinal incomplete-information games have been obtained by Aradillas-López & Gandhi (2016).

Assumption 5 (strategy spaces). Here, the main distinction is whether mixed strategies are allowed or attention is restricted to pure strategies. Identification results in binary-choice games with complete information have been obtained by ruling out mixed strategies. Under symmetry conditions, this leads to a unique prediction for $\sum_{p=1}^{P} Y_p$ in all coexisting equilibria.[2] This result has been exploited by Bresnahan & Reiss (1990; 1991a,b), Berry (1992), and Tamer (2003), and it is discussed in detail by Berry & Tamer (2007, section 2.4). Incomplete-information econometric methods typically assume beliefs that produce a unique optimal action with probability 1 (w.p.1) and therefore lead to pure strategies.

Assumption 6 (multiple solutions and selection mechanism). Multiple equilibria are more prevalent in complete-information games (see Morris & Shin 2003). As a result, an important body of work has been devoted to doing robust inference in complete-information games without any explicit assumptions about the underlying equilibrium selection mechanisms. The list of such studies is extensive, but most are purely econometric papers that include a game as an example. Perhaps the most significant effort to date explicitly devoted to a nontrivial game is by Ciliberto & Tamer (2009). On the other extreme we have papers

[1] Grieco (2014) analyzes a parametric game where unobservables have a particular structure that can combine (i.e., nest) the complete- and incomplete-information frameworks depending on the value of a subset of parameters.

[2] Ciliberto & Tamer (2009) allow for asymmetries, and ruling out mixed strategies does not lead to a unique prediction for $\sum_{p=1}^{P} Y_p$ in all coexisting equilibria.

with an explicit model of equilibrium selection, and one of the most notable complete-information examples is provided by Bajari et al. (2010b).

In econometric studies of incomplete-information games with equilibrium behavior, a commonly made assumption is that the underlying selection mechanism is degenerate, so that it chooses a unique equilibrium w.p.1, or, in other words, the data come from a single equilibrium. This assumption can often help identify the remaining parameters of the model (a result that does not hold in complete-information games). This type of assumption has led to identification results, for example, by Seim (2006), Pesendorfer & Schmidt-Dengler (2008), Bajari et al. (2010a), and Aradillas-López (2012). Studies that do not assume a degenerate selection mechanism include those by de Paula & Tang (2012), Aradillas-López & Gandhi (2016), and Xiao (2018). Examples that include an explicit model of equilibrium selection include work by Ackerberg & Gowrisankaran (2006) and Sweeting (2009). In many instances, the presence of multiple equilibria has identification power, for example, to infer the direction of strategic interaction (de Paula & Tang 2012, Aradillas-López & Gandhi 2016) or the parameters of the model (Ackerberg & Gowrisankaran 2006, Sweeting 2009). Testing for the presence of multiple equilibria has been studied, for example, by Otsu et al. (2016), Hahn et al. (2017), and Marcoux (2018). A survey of econometric methods with multiple equilibria in games is provided by de Paula (2013). Nonequilibrium inference typically makes no assumptions about the selection mechanism but focuses on doing robust inference involving the rest of the parameters (payoffs, beliefs).

Assumption 7 (payoff functions). Methods can be classified according to the parametric assumptions made about payoff functions. The list of studies that parametrize payoffs is vast and includes the majority of existing work. Nonparametric payoff functions are considered, for example, by Aradillas-López (2011, 2019), de Paula & Tang (2012), Kline & Tamer (2012), Lewbel & Tang (2015), Aradillas-López & Gandhi (2016), and Liu et al. (2017), among others. In many nonparametric studies, the goal is to test or infer qualitative features of the game, such as the sign of strategic interaction, the information possessed by players, or the presence of multiple equilibria.

3. IDENTIFICATION AND THE INFERENTIAL PROBLEM

The econometric goal is to perform inference on features of the structure of the game, which include payoffs, strategies, and beliefs. Inference here can be understood as the statistical problem of estimating a set of structures that is consistent with the underlying model, given the data observed. In the frequentist approaches reviewed here, the inferential goal is for the estimated set of structures to have the usual desirable asymptotic properties, such as including the true structure with a prespecified probability as the sample size grows.

Inference is dictated by the observable implications of the model and the resulting identification properties of the underlying structure. Thus, identification and inference are inextricably linked. In econometric terms, if the model is such that there is a unique structure consistent with the data, the model is called point identified. Otherwise, if there are multiple such structures, the model is said to be partially identified. Of particular interest in the literature is the ability to perform inference that is robust to the presence of multiple solutions. This refers to inference that relies only on observable implications of the model that are valid regardless of the presence of multiple solutions and regardless of the features of the underlying selection mechanism.

For brevity, in this section we focus on describing observable implications and constructive identification results under different assumptions, while pointing out how they would be used for statistical inference (for example, using moment inequalities). Some econometric details about inferential methods are described in the following sections (e.g., Section 5.1.2).

3.1. Data Observed

The setting we consider is one in which the econometrician observes $i = 1, \ldots, n$ realizations of the game described above. The researcher observes $Y_i = (Y_{p,i})_{i=1}^{n}$ and X_i, where X_i is a collection of covariates that contain information for $(u_{p,i}, \pi_{-p,i}, \sigma_{p,i})_{p=1}^{P}$. The models we review assume that $(Y_i, X_i)_{i=1}^{n}$ is a random sample, with $(Y, X) \sim P_{X,Y}$ and $P_{X,Y}$ nonparametrically identified.

3.2. Identification and Inference with a Unique Solution or an Explicit Selection Mechanism

If the game has a unique solution almost surely, then the selection mechanism becomes redundant and the observable implications of the model are given by Equation 5 with a trivial selection mechanism that chooses the only existing solution. From here, inference can proceed as with any microeconometric model, depending on the specific assumptions made about the joint distribution of observable and unobservable payoff components, and on the parametric assumptions imposed on the remaining components of the model, such as payoff functions.

Alternatively, the game can have multiple solutions but there may exist an aggregate function $T(Y)$ with the property that all coexisting solutions yield a unique prediction for $T(Y)$. For example, under some symmetry conditions this happens in binary-choice games of complete information and NE behavior for $T(Y) = \sum_{p=1}^{P} Y_p$ if mixed strategies are ruled out (see Berry & Tamer 2007, section 2.4). In these cases, inference can proceed by applying Equation 5 to $T(Y)$, and the specific selection mechanism becomes irrelevant.

If multiple solutions can exist but the model is "completed" with an explicit selection mechanism, identification and inference once again would be based on Equation 5 and on the assumptions made about the joint distribution of observable and unobservable payoff components, as well as on the parametric assumptions imposed on the remaining components of the model, such as payoff functions and the selection mechanism.

3.3. Identification and Inference Without a Selection Mechanism

Let $\Psi_p \equiv (u_p, \pi_{-p}, \sigma_p)$ and $\Psi = \{\Psi_p\}_{p=1}^{P} \in \Upsilon$. Without an explicit model for the selection mechanism, the goal is to do robust inference on Ψ based solely on the implications of the solution concept employed.[3] Take any collection of action profiles $A \subseteq \mathcal{S}$ and let

$$\mathcal{R}(A) = \left\{\Psi \colon \Omega^*(y|u) \neq \emptyset \text{ for some } y \in A\right\}, \qquad 6.$$

where $\Omega^*(y|u)$ is as defined in Equation 4. Note that $Y \in A$ only if $\Psi \in \mathcal{R}(A)$.

3.3.1. A notational convention. For a random variable ξ we use the statements "a.e. ξ", "a.s. in ξ" or "w.p.1. in ξ" interchangeably to refer to an event that is satisfied "for almost every realization of ξ," "almost surely in ξ," or "with probability 1 in ξ."

3.3.2. Parameters and the identified set. The identified set is the collection of all $\Psi \in \Upsilon$ that could have generated the data observed for some selection mechanism. Suppose $(Y, \Psi)|X \sim \mathcal{P}_X \in \mathscr{P}$ (a space of distributions). Any particular $\mathcal{P}_X$ reflects implicit properties about the

[3]One of the first papers to formalize this notion in equilibrium models was by Jovanovic (1989).

underlying selection mechanism, but the latter is not explicitly modeled. The space $\mathscr{P}$ implicitly restricts the class of selection mechanisms allowed. Since $P_{Y,X}$ is nonparametrically identified, the joint distribution of (Y, X, Ψ) is completed by $\mathcal{P}_X$, and we can describe the unknown parameters of the model as $(\Psi, \mathcal{P}_X) \in \Theta$, where $\Theta = \{(\Psi, \mathcal{P}_X)\colon \Psi \in \Upsilon, \mathcal{P}_X \in \mathscr{P}\}$ denotes the parameter space. Using the above definition for $\mathcal{R}$, we obtain

$$Pr(Y \in A|X) \leq \mathcal{P}_X(\Psi \in \mathcal{R}(A)|X) \quad \forall A \subseteq \mathcal{S}, \text{ a.e. } X. \tag{7}$$

The identified set of parameters can be defined from the above condition. However, it can be characterized in an alternative way. Note that for any set, $\mathcal{C} \subseteq \Upsilon$, $\mathcal{R}(Y) \subseteq \mathcal{C}$ implies $\Psi \in \mathcal{C}$ [since $\Psi \in \mathcal{R}(Y)$ a.s.]. The identified set can be defined as

$$\Theta_I = \left\{(\Psi, \mathcal{P}_X) \in \Theta\colon \mathcal{P}_X\left(\mathcal{R}(Y) \subseteq \mathcal{C}|X\right) \leq \mathcal{P}_X\left(\Psi \in \mathcal{C}|X\right) \ \ \forall \mathcal{C} \subseteq \Upsilon, \text{ a.e. } X\right\}. \tag{8}$$

This definition is sharp, meaning that a parameter is in Θ_I if and only if there exists an underlying selection mechanism (not necessarily unique) that makes it consistent with the data observed. The characterization in Equation 8 may be infeasible in practice because it involves all possible subsets in Υ. Let us prespecify a class of subsets $\overline{\mathcal{C}} \subset \Theta$ and define

$$\Theta_I^{\overline{\mathcal{C}}} = \left\{(\Psi, \mathcal{P}_X) \in \Theta\colon \mathcal{P}_X\left(\mathcal{R}(Y) \subseteq \mathcal{C}|X\right) \leq \mathcal{P}_X\left(\Psi \in \mathcal{C}|X\right) \ \ \forall \mathcal{C} \subseteq \overline{\mathcal{C}}, \text{ a.e. } X\right\}.$$

Note that $\Theta_I \subseteq \Theta_I^{\overline{\mathcal{C}}}$. A definition such as $\Theta_I^{\overline{\mathcal{C}}}$ can therefore be conservative but computationally feasible to implement. An important body of econometric work has been devoted to the issue of feasible sharp inference, aimed at characterizing the smallest class of sets $\overline{\mathcal{C}}$ such that $\Theta_I = \Theta_I^{\overline{\mathcal{C}}}$. Such a class is often referred to as the class of core-determining sets (Galichon & Henry 2011). Sharp inference methods have used results from random set theory (Beresteanu & Molinari 2008, Beresteanu et al. 2011, Chesher & Rosen 2017) and optimal transportation theory (Galichon & Henry 2011).

3.3.3. Moment inequality characterizations of the identified set.

Inference can be performed bypassing entirely the selection mechanism. Instead of $\mathcal{P}_X$ (the joint distribution of $(Y, \Psi)|X$), we can focus merely on $\mathcal{Q}_X$, the distribution of $\Psi|X$, and define $\widetilde{\Theta}_I$ as

$$\widetilde{\Theta}_I = \left\{(\Psi, \mathcal{Q}_X) \in \Theta\colon Pr(Y \in A|X) \leq \mathcal{Q}_X(\Psi \in \mathcal{R}(A)|X) \quad \forall A \in \mathcal{S}, \text{ a.e. } X\right\}.$$

Clearly, $\Theta_I \subseteq \widetilde{\Theta}_I$, but $\widetilde{\Theta}_I$ may include Ψ's that are excluded from Θ_I. Inferential methods based on moment inequalities can focus only on a class of subsets in $\mathcal{S}$. For example, in discrete games, we could focus only on the class of singletons in $\mathcal{S}$ described by

$$\widetilde{\Theta}_I^{\mathcal{S}} = \left\{(\Psi, \mathcal{Q}_X) \in \Theta\colon Pr(Y = y|X) \leq \mathcal{Q}_X(\Psi \in \mathcal{R}(y)|X) \quad \forall y \in \mathcal{S}, \text{ a.e. } X\right\}.$$

Econometric inference with moment inequalities has been an area of active research, and game-theoretic models have been among the most important applications. Examples include work by Chernozhukov et al. (2007, 2013), Andrews & Jia-Barwick (2010), Andrews & Soares (2010), Bugni (2010), Romano & Shaikh (2010), Chetverikov (2012), Andrews & Shi (2013), Armstrong (2014, 2015), Romano et al. (2014), and Pakes et al. (2015).

3.3.4. Using a superset for the set of solutions $\mathcal{R}$.

In some cases, a full characterization of the set $\mathcal{R}$ (defined in Equation 6) may require finding all existing solutions to the game, or it may necessitate stronger assumptions about payoff functions (e.g., parametrization) than the researcher is willing to make. In such instances, it may be possible to characterize a superset $\overline{\mathcal{R}}$ that is feasible

Table 1 Matrix form game

	$Y_2 = 1$	$Y_2 = 0$
$Y_1 = 1$	$X_1'\beta_1 + \Delta_1 - \varepsilon_1, X_2'\beta_2 + \Delta_2 - \varepsilon_2$	$X_1'\beta_1 - \varepsilon_1, 0$
$Y_1 = 0$	$0, X_2'\beta_2 - \varepsilon_2$	$0, 0$

to compute under the specific assumptions of the model, with the property that $\mathcal{R}(A) \subseteq \overline{\mathcal{R}}(A)$ for all A. From here, any of the previous characterizations of the identified set can be constructed replacing $\mathcal{R}$ with $\overline{\mathcal{R}}$. This approach is taken, for example, by Aradillas-López (2011), de Paula & Tang (2012), Kline & Tamer (2012), and Aradillas-López & Gandhi (2016).

4. 2 × 2 GAMES

Binary choice games with $Y_p \in \{0, 1\}$ have been studied extensively, particularly to model simultaneous participation or entry decisions. Some examples include work by Bjorn & Vuong (1984), Bresnahan & Reiss (1990), Berry (1992), Tamer (2003), Ciliberto & Tamer (2009), Sweeting (2009), Aradillas-López (2010), de Paula & Tang (2012), and Kline & Tamer (2012). Berry & Reiss (2007) and Berry & Tamer (2007) provide surveys of entry games.

The canonical example of binary-choice games in econometrics is the 2×2 case. This was the focus of the paper by Bjorn & Vuong (1984) that pioneered the econometric analysis of games and was a contribution well ahead of its time. Consider the matrix form game in **Table 1**. $\gamma \equiv (\beta_1, \beta_2, \Delta_1, \Delta_2)$ are unknown parameters. $X \equiv (X_1, X_2)$ and $\varepsilon \equiv (\varepsilon_1, \varepsilon_2)$ are nonstrategic payoff shifters. X is observed by the econometrician, but ε is not. The space of payoff functions in this parametric model is indexed by $\gamma \in \Gamma$ (the parameter space).

Assume for simplicity that $\Delta_p \leq 0$, corresponding to strategic substitutes. We also maintain that $\varepsilon|X$ is jointly continuously distributed with unbounded support $\mathbb{R}^2$. We describe inference of this game under the cases of complete and incomplete information, and in each instance we consider two alternative solution concepts: NE and iterated dominance.

4.1. Inference in the Complete-Information Case

Suppose the true value of γ and the realizations of X and ε are observed by both players.

4.1.1. Nash equilibrium behavior. A strategy profile (σ_1, σ_2) is an NE in this game if $u_p(\sigma_p, \sigma_{-p}) \geq u_p(y_p, \sigma_{-p})$ for $y_p \in \{0, 1\}$.

4.1.1.1. Pure-strategy Nash equilibrium. For a given (X, γ) and any profile $y = (y_1, y_2)$, let $\mathcal{R}_{\text{PSNE}}(y|X, \gamma)$ denote the region of values of $(\varepsilon_1, \varepsilon_2)$ such that y is a pure-strategy NE (PSNE). The regions are as follows:

$$\mathcal{R}_{\text{PSNE}}(1, 1|X, \gamma) = \left\{(\varepsilon_1, \varepsilon_2)\colon X_1'\beta_1 + \Delta_1 - \varepsilon_1 \geq 0, X_2'\beta_2 + \Delta_2 - \varepsilon_2 \geq 0\right\},$$

$$\mathcal{R}_{\text{PSNE}}(1, 0|X, \gamma) = \left\{(\varepsilon_1, \varepsilon_2)\colon X_1'\beta_1 - \varepsilon_1 \geq 0, X_2'\beta_2 + \Delta_2 - \varepsilon_2 \leq 0\right\},$$

$$\mathcal{R}_{\text{PSNE}}(0, 1|X, \gamma) = \left\{(\varepsilon_1, \varepsilon_2)\colon X_1'\beta_1 + \Delta_1 - \varepsilon_1 \leq 0, X_2'\beta_2 - \varepsilon_2 \geq 0\right\}, \text{ and}$$

$$\mathcal{R}_{\text{PSNE}}(0, 0|X, \gamma) = \left\{(\varepsilon_1, \varepsilon_2)\colon X_1'\beta_1 - \varepsilon_1 \leq 0, X_2'\beta_2 - \varepsilon_2 \leq 0\right\}.$$

4.1.1.2. Mixed-strategy Nash equilibrium. A strategy profile (σ_1, σ_2) is a mixed-strategy NE if and only if $u_1(1, \sigma_2) = u_1(0, \sigma_2)$ and $u_2(1, \sigma_1) = u_2(0, \sigma_1)$. In this case, we obtain

$$X_1'\beta_1 - \varepsilon_1 + \sigma_2 \cdot \Delta_1 = 0 \quad \text{and} \quad X_2'\beta_2 - \varepsilon_2 + \sigma_1 \cdot \Delta_2 = 0.$$

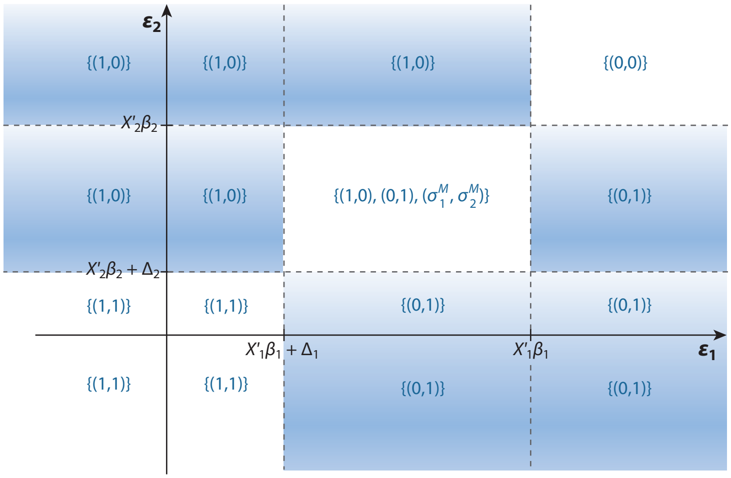

Figure 1

Nash equilibrium regions for the complete-information game.

This yields

$$\sigma_1^M(1) = \frac{X_2'\beta_2 - \varepsilon_2}{-\Delta_2} \quad \text{and} \quad \sigma_2^M(1) = \frac{X_1'\beta_1 - \varepsilon_1}{-\Delta_1}. \tag{9}$$

We have $0 < \sigma_1(1) < 1$ and $0 < \sigma_2(1) < 1$ if and only if $X_1'\beta_1 + \Delta_1 < \varepsilon_1 < X_1'\beta_1$ and $X_2'\beta_2 + \Delta_2 < \varepsilon_2 < X_2'\beta_2$. Thus, we find that

$$\mathcal{R}_{\text{MSNE}}(\sigma_1^M, \sigma_2^M | X, \gamma) = \left\{(\varepsilon_1, \varepsilon_2)\text{: } X_1'\beta_1 + \Delta_1 < \varepsilon_1 < X_1'\beta_1, X_2'\beta_2 + \Delta_2 < \varepsilon_2 < X_2'\beta_2\right\}$$

is the region of mixed-strategy NE. **Figure 1** shows the NE regions.

Define $\mathcal{R}_{\text{NE}}(y|X, \gamma)$ as the region of values of $(\varepsilon_1, \varepsilon_2)$ such that there exists an NE where y can be chosen with nonzero probability. We have

$$\mathcal{R}_{\text{NE}}(y|X, \gamma) = \mathcal{R}_{\text{PSNE}}(y|X, \gamma) \cup \mathcal{R}_{\text{MSNE}}(\sigma_1^M, \sigma_2^M | X, \gamma).$$

Therefore, we find that

$$Pr(Y = y|X) \leq Pr(\varepsilon \in \mathcal{R}_{\text{NE}}(y|X, \gamma)|X) \ \forall y, \ \text{a.e. } X.$$

If we assume $\varepsilon|X \sim G(\cdot|X, \rho)$ (a parametric family of distributions) and we let $\theta \equiv (\gamma, \rho) \in \Theta$, with

$$\int_{\varepsilon \in \mathcal{R}_{NE}(y|X,\gamma)} dG(\varepsilon|X, \rho) \equiv H_{\text{NE}}(y|X, \theta),$$

we can perform inference on θ based on the moment inequalities implied by the model. Define $\overline{\Theta}_I$ as

$$\overline{\Theta}_I = \left\{\theta \in \Theta\text{: } Pr(Y = y|X) \leq H_{\text{NE}}(y|X, \theta) \quad \forall y, \ \text{a.e. } X\right\}.$$

A sharp characterization of the identified set can proceed in the manner described in Section 3.3.2. Suppose $(Y, \varepsilon)|X \sim \mathcal{P}_X \in \mathscr{P}$ (a class of distributions). The parameters of the game are

$(\gamma, \mathcal{P}_X) \in \Theta$, and the identified set can be described as

$$\Theta_I = \left\{(\gamma, \mathcal{P}_X) \in \Theta \colon \mathcal{P}_X(\mathcal{R}_{\mathrm{NE}}(Y|X,\gamma) \subseteq \mathcal{C}|X) \leq \mathcal{P}_X(\varepsilon \in \mathcal{C}|X)\ \forall\, \mathcal{C} \subset \mathbb{R}^2,\ \text{a.e. } X\right\}.$$

4.1.1.3. *Ruling out mixed strategies.* As **Figure 1** shows, if we rule out mixed strategies, every coexisting NE predicts the same value for $Y_1 + Y_2$. We have

$$Pr(Y_1 + Y_2 = 0|X) = Pr\,(\varepsilon \in \mathcal{R}_{\mathrm{PSNE}}((0,0)|X,\gamma)|X),$$

$$Pr(Y_1 + Y_2 = 1|X) = Pr\,(\varepsilon \in \mathcal{R}_{\mathrm{PSNE}}((1,0)|X,\gamma) \cup \mathcal{R}_{\mathrm{PSNE}}((0,1)|X,\gamma)|X),\ \text{and}$$

$$Pr(Y_1 + Y_2 = 2|X) = Pr\,(\varepsilon \in \mathcal{R}_{\mathrm{PSNE}}((1,1)|X,\gamma)|X).$$

Inference for the parameters of the model can be performed from here. This result extends beyond two players under some symmetry conditions of strategic effects and has been widely used for identification purposes (see Berry & Tamer 2007, section 2.4).

4.1.2. Iterated dominance behavior. NE behavior presupposes that players' beliefs are correct. A weaker requirement that is still consistent with rationality and expected-payoff maximization is to allow players to have incorrect beliefs but assume that these beliefs assign zero probability to dominated strategies. Iteratively removing dominated strategies in this fashion leads to the solution concept of iterated dominance. The following analysis was proposed by Aradillas-López & Tamer (2008). The steps of iterated dominance are described below.

Step 1: Eliminate all dominated strategies in $\mathcal{S}$. Define $\mathcal{S}^1$ to be the strategy profiles that remain.

Step 2: Eliminate all strategies that are not best responses to beliefs concentrated on $\mathcal{S}^1$. Define $\mathcal{S}^2$ to be the strategy profiles that remain.

⋮

Step k: Eliminate all strategies that are not best responses to beliefs concentrated on $\mathcal{S}^{k-1}$. Define $\mathcal{S}^k$ to be the strategy profiles that remain.

⋮

This process stops when there are no more strategies left to eliminate. The set of strategies $\mathcal{S}^*$ that survive represents the rationalizable strategies.[4] By construction, all NE strategies must be rationalizable and, therefore, contained in $\mathcal{S}^k$ for any k. In the 2×2 game convergence is achieved after at most two steps.

Step 1: Dominated strategies are described as follows:

$$Y_p = 1 \text{ is dominated if and only if } X_p'\beta_p - \varepsilon_p < 0 \text{ (i.e., } \varepsilon_p > X_p'\beta_p).$$

$$Y_p = 0 \text{ is dominated if and only if } X_p'\beta_p + \Delta_p - \varepsilon_p > 0 \text{ (i.e., } \varepsilon_p < X_p'\beta_p + \Delta_p).$$

Step 2: If there are no dominated strategies, the process ends in Step 1. Otherwise,

If $Y_p = 0$ is dominated:

$$Y_{-p} = 1 \text{ is dominated if and only if } X_{-p}'\beta_{-p} + \Delta_p - \varepsilon_{-p} < 0 \text{ (i.e., } \varepsilon_{-p} > X_{-p}'\beta_{-p} + \Delta_p).$$

$$Y_{-p} = 0 \text{ is dominated if and only if } X_{-p}'\beta_{-p} + \Delta_p - \varepsilon_{-p} > 0 \text{ (i.e., } \varepsilon_{-p} < X_{-p}'\beta_{-p} + \Delta_p).$$

[4]The concept of rationalizability and its relationship with iterated dominance were developed and analyzed by Bernheim (1984) and Pearce (1984).

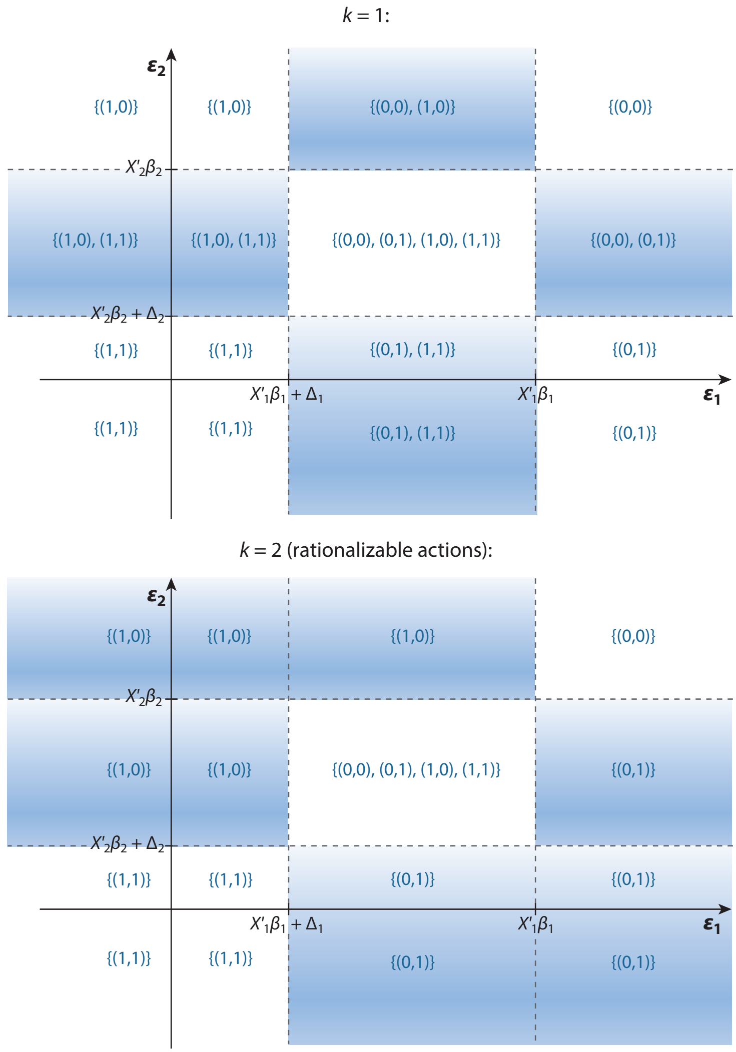

Figure 2

Action profiles that survive k rounds of iterated dominance.

If $Y_p = 1$ is dominated:

$Y_{-p} = 1$ is dominated if and only if $X'_{-p}\beta_{-p} - \varepsilon_{-p} < 0$ (i.e., $\varepsilon_{-p} > X'_{-p}\beta_{-p}$).

$Y_{-p} = 0$ is dominated if and only if $X'_{-p}\beta_{-p} - \varepsilon_{-p} > 0$ (i.e., $\varepsilon_{-p} < X'_{-p}\beta_{-p}$).

Figure 2 shows the action profiles that survive k rounds of iterated dominance, with $k = 1, 2$. Let $\mathcal{S}^k(X, \varepsilon, \gamma)$ denote the set of strategies that survive k iterated dominance steps and define

$$\mathcal{R}^k_{\text{ID}}(y|X,\gamma) = \big\{(\varepsilon_1, \varepsilon_2) \colon y \in \mathcal{S}^k(X, \varepsilon, \gamma)\big\}.$$

Then, under the assumption that strategies survive k steps of iterated dominance, we obtain

$$Pr(Y = y|X) \leq Pr(\varepsilon \in \mathcal{R}^k_{\text{ID}}(y|X,\gamma)|X) \;\forall y, \text{ a.e. } X.$$

Assuming iterated dominance instead of NE behavior, the sets $\overline{\Theta}_I$ and Θ_I described in the NE case would be redefined by replacing $\mathcal{R}_{\text{NE}}$ with $\mathcal{R}^k_{\text{ID}}$.

4.2. Inference in the Incomplete-Information Case

Suppose now that the realization of a subset of payoff shifters is only private information. For simplicity, suppose $X \equiv (X_1, X_2)$ is observed by both players, but ε_p is only privately observed by player p. Econometric analysis of incomplete-information games typically starts by assuming that beliefs are conditioned on a certain information observed by players. Let W_p denote the information used by player p to condition their beliefs. Assume throughout that $X \in W_p$ (since both players observe X). Let $\pi_{-p}(W_p)$ denote p's subjective beliefs for the probability that $Y_{-p} = 1$. Player p's expected payoff of choosing $Y_p = 1$ is given by

$$\overline{u}_p(1, \pi_{-p}, X_p, \varepsilon_p) = X_p'\beta_p + \Delta_p \cdot \pi_{-p}(W_p) - \varepsilon_p, \tag{10.}$$

and $\overline{u}_p(0, \pi_{-p}, X_p, \varepsilon_p) = 0$. We assume that players follow a threshold-crossing decision rule:[5]

$$Y_p = \mathbb{1}\{X_p'\beta_p + \Delta_p \cdot \pi_{-p}(W_p) - \varepsilon_p \geq 0\}.$$

Econometric inference in incomplete-information games typically proceeds by making more precise assumptions about W_p. Given that X is observed by both players, the prototypical econometric study would rely on the following two assumptions in this case:

Assumption 8 (conditional independence of players' private information). Conditional on X, we have $\varepsilon_1 \perp \varepsilon_2$.

Assumption 9 (information used by players). $W_p = X$ for both $p = 1, 2$.

Thus, expected payoffs in Equation 10 are of the form

$$\overline{u}_p(1, \pi_{-p}, X_p, \varepsilon_p) = X_p'\beta_p + \Delta_p \cdot \pi_{-p}(X) - \varepsilon_p. \tag{11.}$$

Conditional independence of private information is a common assumption in the econometric analysis of incomplete-information games, but recent efforts have been made to relax this restriction (see the papers cited in Section 2). Most existing methods can be extended to cases where $W_p = (X, \xi_p)$ (where ξ_p is informative for ε_{-p}) as long as ξ_p is observable to the econometrician. For simplicity we consider the case $W_p = X$.

4.2.1. Bayesian Nash equilibrium behavior.

Suppose $\varepsilon|X \sim G_X$ and $\varepsilon_p|X \sim G_{p,X}(\cdot)$. For a given $\pi \equiv (\pi_1, \pi_2) \in [0, 1]^2$, let

$$\Lambda(\pi|X, \gamma, G_X) = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} - \begin{pmatrix} G_{1,X}(X_1'\beta_1 + \Delta_1 \cdot \pi_2) \\ G_{2,X}(X_2'\beta_2 + \Delta_2 \cdot \pi_1) \end{pmatrix}. \tag{12.}$$

Given (X, γ, G_X), a Bayesian Nash equilibrium (BNE) is a pair of beliefs π^* for players 1 and 2 that solves the fixed point condition

$$\Lambda(\pi|X, \gamma, G_X) = 0. \tag{13.}$$

[5]This can be done without loss of generality under the assumption that $\varepsilon_p|X_p, W_p$ is continuously distributed with unbounded support, so that $Pr(X_p'\beta_p + \Delta_p \cdot \pi_{-p}(W_p) - \varepsilon_p = 0) = 0$.

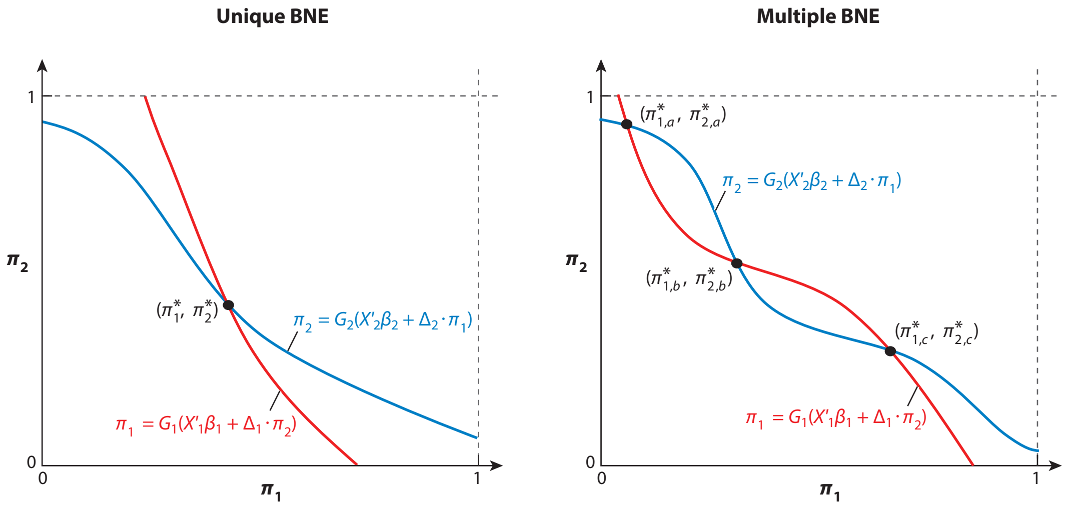

Figure 3

An illustration of Bayesian Nash equilibria (BNE).

Let $g_{p,X}$ denote the density function corresponding to the distribution $G_{p,X}$, and let

$$\nabla_\pi \Lambda(\pi \mid X, \gamma, G_X) = \begin{pmatrix} 1 & -\Delta_1 \cdot g_{1,X}(X_1'\beta_1 + \Delta_1 \cdot \pi_2) \\ -\Delta_2 \cdot g_{2,X}(X_2'\beta_2 + \Delta_2 \cdot \pi_1) & 1 \end{pmatrix}$$

denote the Jacobian of the BNE system with respect to π. A solution π^* to the BNE system is regular if $\det(\nabla_\pi \Lambda(\pi^* | X, \gamma, G_X)) \neq 0$. That is, we find that

$$1 - \Delta_1 \cdot \Delta_2 \cdot g_{1,X}(X_1'\beta_1 + \Delta_1 \cdot \pi_2) \cdot g_{2,X}(X_2'\beta_2 + \Delta_2 \cdot \pi_1) \neq 0. \qquad 14.$$

By the implicit function theorem, regular BNE are locally unique, well-defined functionals of (X, γ, G_X).

4.2.1.1. Existence and multiplicity of Bayesian Nash equilibria. Existence of a solution to Equation 13 follows from Brouwer's fixed point theorem. Sufficient conditions for uniqueness of a solution can be obtained by verifying the Gale-Nikaido conditions (Gale & Nikaido 1965). If all the principal minors of $\nabla_\pi \Lambda(\pi | X, \gamma, G_X)$ are positive, then the BNE solution to Equation 13 will be unique. This will be satisfied if

$$1 - \Delta_1 \cdot \Delta_2 \cdot g_{1,X}(X_1'\beta_1 + \Delta_1 \cdot \pi_2) \cdot g_{2,X}(X_2'\beta_2 + \Delta_2 \cdot \pi_1) > 0. \qquad 15.$$

While Equation 14 ensures local uniqueness, Equation 15 ensures global uniqueness.[6] Note that Equation 15 is immediately satisfied if $\Delta_1 \cdot \Delta_2 \leq 0$, but it may not hold if $\Delta_1 \cdot \Delta_2 > 0$. Also note that if X_1 or X_2 contains an element with rich-enough support, then by restricting attention to regions where such a covariate is sufficiently negative or sufficiently positive, we can make $g_{1,X}(X_1'\beta_1 + \Delta_1 \cdot \pi_2) \cdot g_{2,X}(X_2'\beta_2 + \Delta_2 \cdot \pi_1)$ sufficiently small that Equation 15 is satisfied and uniqueness is achieved. This idea was explored by Aradillas-López (2010). **Figure 3** illustrates cases with unique BNE and multiple BNE.

[6]Equation 15 is a sufficient but not necessary condition for uniqueness.

4.2.1.2. Identified set. For a given (X, γ, G_X), let $\Pi_{\text{BNE}}(X, \gamma, G_X)$ denote the set of all solutions to the BNE system in Equation 13. Let

$$\mathcal{R}_{\text{BNE},p}(1|X,\gamma,G_X) = \{\varepsilon_p\colon X_{-p}'\beta_p + \Delta_p\pi_{-p}^* - \varepsilon_p \geq 0 \text{ for some } \pi^* \in \Pi_{\text{BNE}}(X,\gamma,G_X)\};$$

$$\mathcal{R}_{\text{BNE},p}(0|X,\gamma,G_X) = \{\varepsilon_p\colon X_{-p}'\beta_p + \Delta_p\pi_{-p}^* - \varepsilon_p < 0 \text{ for some } \pi^* \in \Pi_{\text{BNE}}(X,\gamma,G_X)\}; \text{ and}$$

$$\mathcal{R}_{\text{BNE}}((y_1,y_2)|X,\gamma,G_X) = \mathcal{R}_{\text{BNE},1}(y_1|X,\gamma,G_X) \times \mathcal{R}_{\text{BNE},2}(y_2|X,\gamma,G_X).$$

Note that

$$Pr(Y = y|X) \leq Pr(\varepsilon \in \mathcal{R}_{\text{BNE}}(y|X,\gamma,G_X))\ \forall y, \text{a.e. } X.$$

The identified set can be characterized from here in the manner described previously. Moment inequalities can be obtained as follows. For a given (X, γ, G_X), let

$$\underline{\pi}_p^*(X,\gamma,G_X) = \min\{\pi_p^* \in \Pi_{\text{BNE}}(X,\gamma,G_X)\}, \quad \overline{\pi}_p^*(X,\gamma,G_X) = \max\{\pi_p^* \in \Pi_{\text{BNE}}(X,\gamma,G_X)\}.$$

Note that

$$\mathbb{1}\{X_p'\beta_p + \Delta_p \cdot \overline{\pi}_{-p}^*(X,\gamma,G_X) - \varepsilon_p \geq 0\} \leq Y_p \leq \mathbb{1}\{X_p'\beta_p + \Delta_p \cdot \underline{\pi}_{-p}^*(X,\gamma,G_X) - \varepsilon_p \geq 0\} \text{ a.e. } X.$$

Moment inequalities can be obtained from here:

$$E[Y_p|X] \leq G_X(X_p'\beta_p + \Delta_p \cdot \underline{\pi}_{-p}^*(X,\gamma,G_X)) \text{ a.e. } X;$$

$$E[Y_p|X] \geq G_X(X_p'\beta_p + \Delta_p \cdot \overline{\pi}_{-p}^*(X,\gamma,G_X)) \text{ a.e. } X.$$

Inference can proceed from here in the ways described previously.

4.2.1.3. Inference assuming a degenerate equilibrium selection mechanism. In incomplete-information games, point identification can be obtained by assuming that the underlying equilibrium selection mechanism $\mathcal{M}$ is degenerate (i.e., it does not randomize across existing BNE), without having to assume which equilibrium is chosen. This is an important distinction with complete-information games, where simply assuming a degenerate selection mechanism is not enough: One would need to assume which equilibrium is chosen. To illustrate, suppose $\Pi_{\text{BNE}}(X, \gamma, G_X)$ is a finite set (e.g., suppose the regularity condition in Equation 14 is satisfied). Then, we obtain

$$E[Y_p|X] = E_X\left[\sum_{\pi_j \in \Pi_{\text{BNE}}(X,\gamma,G_X)} \mathbb{1}\{X_p'\beta_p + \Delta_p\pi_{-p,j}(X,\gamma,G_X) - \varepsilon_p \geq 0\} \cdot Pr\left(\mathcal{M} \text{ selects } \pi_j|X,\varepsilon_p\right)\middle| X\right]. \tag{16}$$

Suppose $\mathcal{M}$ selects one BNE w.p.1. We say in this case that the data is generated from a single equilibrium. Let π^* denote this BNE. The previous equation becomes

$$E[Y_p|X] = E\left[\mathbb{1}\{X_p'\beta_p + \Delta_p\pi_{-p}^*(X,\gamma,G_X) - \varepsilon_p \geq 0\}\middle| X\right] = G_X(X_p'\beta_p + \Delta_p\pi_{-p}^*(X,\gamma,G_X)) \quad \text{for } p = 1, 2.$$

But since π^* is a BNE solution, it follows that

$$E[Y_p|X] = G_X\left(X_p'\beta_p + \Delta_p E[Y_{-p}|X]\right) \quad \text{for } p = 1, 2. \tag{17}$$

Since $E[Y_p|X]$ is nonparametrically identified, inference for the payoff parameters γ can proceed from here using a two-step procedure. In the first step we nonparametrically estimate $E[Y_p|X]$, and in the second step we plug them into the above equation to estimate the parameters of the model.[7] A variety of econometric methods can be applied, including those that allow for a nonparametrically specified G_X (see, for example, Powell et al. 1989, Ahn & Manski 1993, Ichimura 1993, Klein & Spady 1993, Ahn 1995). Examples of econometric studies that assume a degenerate equilibrium selection mechanism include work by Seim (2006), Pesendorfer & Schmidt-Dengler (2008), Bajari et al. (2010a), and Aradillas-López (2012).[8] Note that counterfactual analysis would still require assumptions about which equilibrium is selected.

4.2.1.4. Dropping the conditional independence assumption. Assuming independence across players' private information is a strong limitation shared by most papers in this literature. In general, this assumption has been dropped by either (*a*) assuming a fully parametric joint distribution for privately observed shocks (e.g., Xu 2014) or (*b*) assuming that the source of correlation is a shock with just a finite number of support points (e.g, Marcoux 2018). A completely different approach was proposed by Aradillas-López (2010), who assumed beliefs to be of the type $E[Y_{-p}|X, Y_p]$ instead of $E[Y_{-p}|X, \varepsilon_p]$. This allows, for example, to treat the joint distribution of $(\varepsilon_1, \varepsilon_2)$ nonparametrically without restricting their joint support. Whether beliefs are of the form $E[Y_{-p}|X, Y_p]$ or $E[Y_{-p}|X, \varepsilon_p]$ is testable. More work should be devoted to testing conditional independence of private information.

4.2.2. Iterated dominance behavior. Once again, we can replace BNE with the weaker requirement of iterated dominance, as done by Aradillas-López & Tamer (2008). We maintain that players follow threshold-crossing decision rules

$$Y_p = \mathbb{1}\{X_p'\beta_p + \Delta_p \cdot \pi_{-p}(X) - \varepsilon_p \geq 0\},$$

but we no longer impose the requirement that beliefs solve the BNE conditions. Iterated dominance can restrict the range of possible beliefs. This can be done naturally because beliefs are bounded in the [0, 1] interval.[9] In what follows, let $G_p(\cdot\,|X)$ denote the distribution of $\varepsilon_p|X$.

Step 1: Since $\pi_{-p}(X) \in [0, 1]$ and actions are strategic substitutes, we find that

$$\mathbb{1}\{X_p\beta_p + \Delta_p - \varepsilon_p > 0\} \leq Y_p \leq \mathbb{1}\{X_p\beta_p - \varepsilon_p \geq 0\}.$$

Therefore, we obtain

$$\underbrace{G_p(X_p\beta_p + \Delta_p|X)}_{\underline{\pi}_p^1(X)} \leq Pr(Y_p = 1|X) \leq \underbrace{G_p(X_p\beta_p|X)}_{\overline{\pi}_p^1(X)}.$$

[7]Assuming that the underlying selection mechanism is degenerate and all the data come from the same equilibrium is an almost universal assumption in econometric models of dynamic games (see, for example, Aguirregabiria & Mira 2007, Pakes et al. 2007, Pesendorfer & Schmidt-Dengler 2008).

[8]The approach proposed by Aradillas-López (2010) relies on using, as an inferential range, a subset of values of X where a Gale-Nikaido condition of the type given in Equation 15 is satisfied and therefore the BNE is unique.

[9]The approach we describe here can be generally applied to any game where actions have bounded support. The binary-choice game is the simplest such case, since $Y_p\{0, 1\}$.

Step 2: Eliminate all strategies that are not best responses to beliefs that survive Step 1. It follows that

$$\mathbb{1}\{X_p\beta_p + \Delta_p \cdot \overline{\pi}^1_{-p}(X) - \varepsilon_p > 0\} \leq Y_p \leq \mathbb{1}\{X_p\beta_p + \Delta_p \cdot \underline{\pi}^1_{-p}(X) - \varepsilon_p \geq 0\}.$$

Therefore,

$$\underbrace{G_p(X_p\beta_p + \Delta_p \cdot \overline{\pi}^1_{-p}(X)|X)}_{\underline{\pi}^2_p(X)} \leq Pr(Y_p = 1|X) \leq \underbrace{G_p(X_p\beta_p + \Delta_p \cdot \underline{\pi}^1_{-p}(X)|X)}_{\overline{\pi}^2_p(X)}.$$

⋮

Step k: Eliminate all strategies that are not best responses to beliefs that survive Step $k-1$. It follows that

$$\mathbb{1}\{X_p\beta_p + \Delta_p \cdot \overline{\pi}^{k-1}_{-p}(X) - \varepsilon_p > 0\} \leq Y_p \leq \mathbb{1}\{X_p\beta_p + \Delta_p \cdot \underline{\pi}^{k-1}_{-p}(X) - \varepsilon_p \geq 0\}.$$

Therefore,

$$\underbrace{G_p(X_p\beta_p + \Delta_p \cdot \overline{\pi}^{k-1}_{-p}(X)|X)}_{\underline{\pi}^k_p(X)} \leq Pr(Y_p = 1|X) \leq \underbrace{G_p(X_p\beta_p + \Delta_p \cdot \underline{\pi}^{k-1}_{-p}(X)|X)}_{\overline{\pi}^k_p(X)}.$$

⋮

Unlike the complete-information case, this process can continue indefinitely. Rationalizable choice probabilities (and therefore BNE choice probabilities) are always included in the interval

$$\left[\underline{\pi}^k_{-p}(X), \overline{\pi}^k_{-p}(X)\right]$$

for any k. From here, a characterization of the identified set under k steps of iterated dominance can proceed analogously to the cases previously discussed. **Figure 4** illustrates the iterative procedure for $k = 1, 2, 3$.

5. DISCRETE GAMES WITH RICHER ACTION SPACES: RECENT RESULTS

Many real-world applications cannot be handled by a binary-choice game. Recently, identification and inference results have been obtained in discrete games where the action space is rich but has an ordinal property. Suppose players' actions are real-valued and the action spaces in our normal-form game in Equation 1 are discrete and ordinal, that is,

$$\mathcal{S}_p = \left\{s^1_p, s^2_p, \ldots, s^{M_p}_p\right\}, \tag{18.}$$

with $s^j_p < s^{j+1}_p$ for all j. None of the results we discuss below presupposes precise knowledge of the elements in $\mathcal{S}_p$, relying only on the ordinal nature and the assumption that either (*a*) the upper and lower bounds of $\mathcal{S}_p$ are known (or can be estimated) or (*b*) inference is focused on the interior of $\mathcal{S}_p$. Binary-choice games are always included as a special case of every model we discuss next.

5.1. Inference in Games with Complete Information

We begin by discussing results for complete-information games.

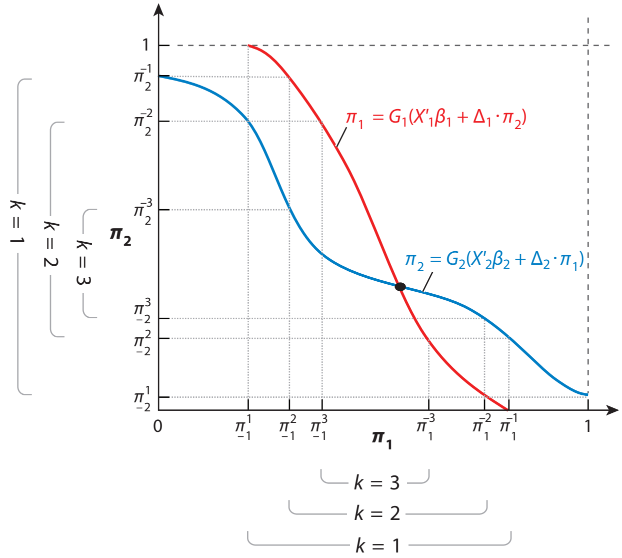

Figure 4

Illustration of iterated dominance in the incomplete-information game.

5.1.1. Nonparametric results. Aradillas-López (2011) derives testable implications for Nash equilibrium outcomes in ordered discrete games with action spaces such as those in Equation 18. The results obtained do not parametrize payoff functions but rely instead only on shape restrictions. The following properties are assumed to hold w.p.1.

Condition 1 (concavity of payoffs). For any $y_{-p} \in \mathcal{S}_{-p}$, we have

$$u_p(s_p^j, y_{-p}) - u_p(s_p^{j-1}, y_{-p}) > u_p(s_p^{j+1}, y_{-p}) - u_p(s_p^j, y_{-p}) \quad \forall s_p^j \in \mathcal{S}_p.$$

Condition 2 (nonincreasing differences). For any $s_p^j \in \mathcal{S}_p$ and $y_{-p}, y'_{-p} \in \mathcal{S}_{-p}$, if $u_p(s_p^j, y_{-p}) \geq u_p(s_p^j, y'_{-p})$, then we have

$$u_p(s_p^{j+1}, y_{-p}) - u_p(s_p^j, y_{-p}) \geq u_p(s_p^{j+1}, y'_{-p}) - u_p(s_p^j, y'_{-p}).$$

Figure 5 illustrates the nonincreasing differences condition. It is reminiscent of properties in supermodular games, and it results in a no-crossing property for payoff functions.

Aradillas-López (2011) assumes complete-information NE behavior without ruling out mixed strategies. Concavity of payoffs and the independent-mixing nature of NE imply that, in any NE, players can randomize across at most two possible actions, and these actions must be adjacent. Fix an action profile $y \equiv (y_p)_{p=1}^P$, where $y_p = s_p^j \in \mathcal{S}_p$ for each p (we write j instead of j_p for notational simplicity). Let

$$S(y) = \left\{ (a_p)_{p=1}^P : a_p \in \{s_p^{j-1}, s_p^j, s_p^{j+1}\} \right\}. \qquad 19.$$

The support of any NE where y is played with positive probability must be a subset of $S(y)$. Next, for each p let[10]

$$\overline{u}_p(\cdot, S(y)) = \max_{y_{-p} \in S(y)} u_p(\cdot, y_{-p}), \quad \text{and} \quad \underline{u}_p(\cdot, S(y)) = \min_{y_{-p} \in S(y)} u_p(\cdot, y_{-p}).$$

[10]Concavity and nonincreasing differences imply that payoff functions do not cross (see **Figure 5**).

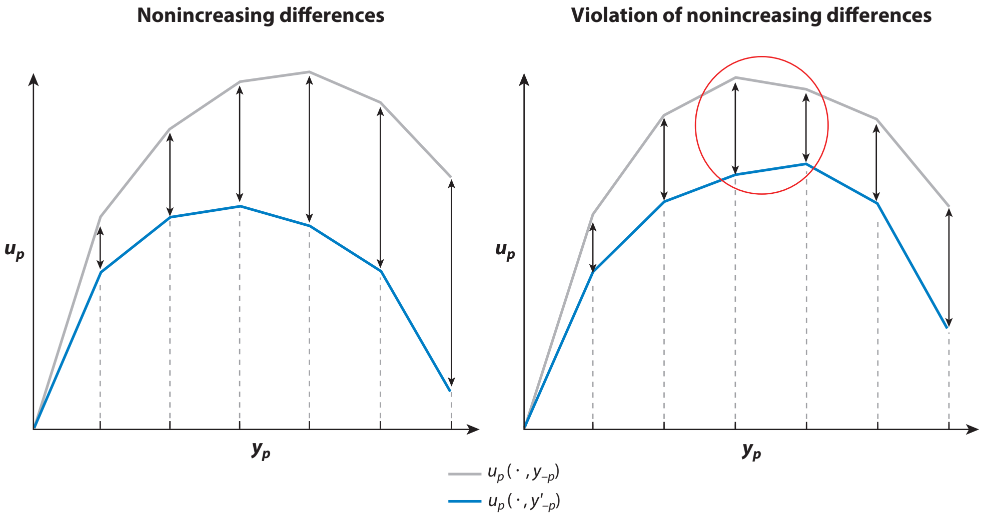

Figure 5

Nonincreasing differences. The red circle illustrates a violation of nonincreasing differences.

By concavity and nonincreasing differences, the action profile y can be played with positive probability in an NE only if

$$\overline{u}_p(s_p^{j-1}, S(y)) < \overline{u}_p(s_p^j, S(y)) \quad \text{and} \quad \underline{u}_p(s_p^j, S(y)) > \underline{u}_p(s_p^{j+1}, S(y)) \quad \forall p.$$

Our shape conditions restrict which NE can coexist. Suppose there exists an NE where $y = (y_p)_{p=1}^P$ is played with positive probability. Now, take another action profile $y' = (y'_p)_{p=1}^P$. Then, there coexists an NE where y' is played with positive probability only if, for each p, one of the following holds:

$$\begin{aligned}
&1. \quad \overline{u}_p(\cdot, S(y')) > \underline{u}_p(\cdot, S(y)) \ \text{ and } \ \underline{u}_p(\cdot, S(y')) < \overline{u}_p(\cdot, S(y));\\
&2. \quad \overline{u}_p(\cdot, S(y')) \leq \underline{u}_p(\cdot, S(y)) \ \text{ and } \ y'_p \leq y_p; \text{ or}\\
&3. \quad \underline{u}_p(\cdot, S(y')) \geq \overline{u}_p(\cdot, S(y)) \ \text{ and } \ y'_p \geq y_p.
\end{aligned} \qquad 20.$$

Let

$$\begin{aligned}
\mathbb{I}_p^a(y, y') &= 1 - \mathbb{1}\{\overline{u}_p(\cdot, S(y')) \leq \underline{u}_p(\cdot, S(y)) \ \text{ and } \ y'_p > y_p\};\\
\mathbb{I}_p^b(y, y') &= 1 - \mathbb{1}\{\underline{u}_p(\cdot, S(y')) \geq \overline{u}_p(\cdot, S(y)) \ \text{ and } \ y'_p < y_p\};\\
\mathbb{I}_p^*(y, y') &= \min\{\mathbb{I}_p(y, y'), \mathbb{I}_p^b(y, y')\}; \qquad \text{and}\\
\mathbb{I}^*(y, y') &= \prod_{p=1}^{P} \mathbb{I}_p^*(y, y').
\end{aligned}$$

Note that $\mathbb{I}^*(y, y')$ is the indicator for the event that at least one of the conditions in Equation 20 is satisfied for each p.

Definition 8 (Nash Equilibrium outcome). We say that $y \equiv (y_p)_{p=1}^P$ is an NE outcome if there exists an NE where y is played with positive probability. Let $\mathcal{E}$ denote the collection of all NE outcomes in the game.

Pick any action profile $y = (y_p)_{p=1}^P$. If we maintain that $Y \equiv (Y_p)_{p=1}^P \in \mathcal{E}$, then

$$\mathbb{1}\{Y = y\} \leq \mathbb{1}\{y \in \mathcal{E}\} \leq \mathbb{I}^*(Y, y) \ \text{ w.p.1.} \qquad 21.$$

If payoffs are completely unrestricted beyond the shape restrictions, $\mathbb{I}^*$ is unobserved and Equation 21 cannot be used directly. Aradillas-López (2011) proposes a way to exploit Equation 21 by assuming the existence of a strategic interaction index, a known function f_p (possibly multi-valued) that captures the direction of strategic interaction.

Definition 9 (existence of a strategic interaction index). For each p there exists a function $f_p : \mathcal{S}_{-p} \to \mathbb{R}^{d_p}$ ($d_p \geq 1$) known to the researcher such that, for any $y_{-p}, y'_{-p} \in \mathcal{S}_{-p}$, the following apply:

1. If $f_p(y'_{-p}) = f_p(y_{-p})$, then $u_p(\cdot, y_{-p}) = u_p(\cdot, y'_{-p})$;
2. If $f_p(y'_{-p}) \geq f_p(y_{-p})$ (element-wise), then $u_p(\cdot, y_{-p}) \leq u_p(\cdot, y'_{-p})$; and
3. If $f_p(y'_{-p}) \ngeq f_p(y_{-p})$ and $f_p(y_{-p}) \ngeq f_p(y'_{-p})$, nothing is implied about the ordinal relationship between $u_p(\cdot, y_{-p})$ and $u_p(\cdot, y'_{-p})$.

Thus, ranking the strategic index f_p enables us to rank payoff functions. Let

$$\mathbb{H}_p(y, y') = \mathbb{1}\big\{ f_p(u_{-p}) \geq f_p(v_{-p}) \ \forall\, u_{-p} \in S(y),\, v_{-p} \in S(y') \big\}.$$

$\mathbb{H}_p(y, y') = 1$ implies $u_p(\cdot, u_{-p}) \leq u_p(\cdot, v_{-p})$ $\forall u_{-p} \in S(y), v_{-p} \in S(y')$. Therefore, we have

$$\mathbb{H}_p(y, y') \leq \mathbb{1}\{\overline{u}_p(\cdot, S(y)) \leq \underline{u}_p(\cdot, S(y'))\},$$

and

$$\begin{aligned}
&\mathbb{I}_p^a(y, y') \leq 1 - \mathbb{H}_p(y', y) \cdot \mathbb{1}\{y'_p > y_p\} \equiv \overline{\mathbb{I}}_p^a(y, y'),\\
&\mathbb{I}_p^b(y, y') \leq 1 - \mathbb{H}_p(y, y') \cdot \mathbb{1}\{y'_p < y_p\} \equiv \overline{\mathbb{I}}_p^b(y, y'),\\
\Longrightarrow\ &\mathbb{I}_p^*(y, y') \leq \min\left\{\overline{\mathbb{I}}_p^a(y, y'), \overline{\mathbb{I}}_p^b(y, y')\right\} \equiv \overline{\mathbb{I}}_p^*(y, y'),\\
\Longrightarrow\ &\mathbb{I}^*(y, y') \leq \prod_{p=1}^P \overline{\mathbb{I}}_p^*(y, y') \equiv \overline{\mathbb{I}}^*(y, y').
\end{aligned}$$

From Equation 21 we obtain

$$\mathbb{1}\big\{Y = y\big\} \leq \mathbb{1}\big\{y \in \mathcal{E}\big\} \leq \overline{\mathbb{I}}^*(Y, y) \quad \forall\, y, \text{ w.p.1.}$$

Let $\mathcal{C}$ denote a collection of outcomes. Then, we obtain

$$\begin{aligned}
\mathbb{1}\{Y \in \mathcal{C}\} \leq \mathbb{1}\{\mathcal{C} \cap \mathcal{E} \neq \emptyset\} &\leq \max_{y \in \mathcal{C}}\{\overline{\mathbb{I}}^*(Y, y)\},\\
\mathbb{1}\{\mathcal{C} \subseteq \mathcal{E}\} &\leq \min_{y \in \mathcal{C}}\{\overline{\mathbb{I}}^*(Y, y)\}.
\end{aligned}$$

The first and the second line relate to the event that some $\mathcal{C}$ contains an NE outcome and that every outcome in $\mathcal{C}$ is an NE outcome, respectively. Let X denote the set of observable covariates by the econometrician. Bounds for the probability of equilibrium outcomes are obtained from the previous inequalities,

$$
\begin{aligned}
Pr(Y = y|x) \leq Pr(y \in \mathcal{E}|X) &\leq E\left[\mathbb{I}^*(Y, y)|X\right], \\
Pr(Y \in \mathcal{C}|X) \leq Pr(\mathcal{C} \cap \mathcal{E} \neq \emptyset|X) &\leq E\left[\max_{y \in \mathcal{C}}\{\overline{\mathbb{I}}^*(Y, y)\}\middle|X\right], \text{ and} \\
Pr(\mathcal{C} \subseteq \mathcal{E}|X) &\leq E\left[\min_{y \in \mathcal{C}}\{\overline{\mathbb{I}}^*(Y, y)\}_{y \in \mathcal{C}}\middle|X\right].
\end{aligned}
\qquad 22.
$$

Since the bounds in Equation 22 are nonparametrically identified, confidence intervals for $Pr(y \in \mathcal{E}|X)$, $Pr(\mathcal{C} \cap \mathcal{E} \neq \emptyset|X)$, and $Pr(\mathcal{C} \subseteq \mathcal{E}|X)$ can be constructed using, for example, the methods described by Imbens & Manski (2004) and Stoye (2009). Aradillas-López (2011) also describes bounds for other probabilities of interest, such as the propensity to select a particular y conditional on being an NE profile.

5.1.2. Parametric results. A parametric ordered-response game of complete information was first analyzed by Davis (2006) under parametrization restrictions that yield a unique PSNE. A much more general model is studied by Aradillas-López & Rosen (2019). Action spaces are as described in Equation 18. Aradillas-López & Rosen (2019) impose restrictions on payoff functions that effectively turn the game into a simultaneous ordered-response model. First, they assume that we can express payoffs as

$$u_p(y_p, y_{-p}, X_p, \varepsilon_p),$$

where X_p is observed by the econometrician while ε_p is an unobserved scalar. Let $X \equiv (X_p)_{p=1}^P$ and $\varepsilon \equiv (\varepsilon_p)_{p=1}^P$. Strict concavity of u_p with respect to y_p is maintained. Aradillas-López & Rosen (2019) then impose an assumption about how payoff functions shift with ε_p.

Assumption 10 (increasing differences in (y_p, ε_p)). For any $y_{-p} \in \mathcal{S}_{-p}$ and a.e. X_p, the following holds: If $\varepsilon_p' > \varepsilon_p$ and $y_p' > y_p$, then

$$u_p(y_p', y_{-p}, X_p, \varepsilon_p) - u_p(y_p, y_{-p}, X_p, \varepsilon_p) < u_p(y_p', y_{-p}, X_p, \varepsilon_p') - u_p(y_p, y_{-p}, X_p, \varepsilon_p').$$

Figure 6 illustrates the increasing differences property.

Fix X_p, ε_p and take any y_{-p}. Let $y_p^*(y_{-p}, X_p, \varepsilon_p)$ denote p's best response. The key implication of the increasing-differences shape restriction is the following: There exists a sequence of nonoverlapping thresholds

$$\varepsilon_p^*\left(s_p^1, y_{-p}, X_p\right) < \varepsilon_p^*\left(s_p^2, y_{-p}, X_p\right) < \cdots < \varepsilon_p^*(s_p^{M_p}, y_{-p}, X_p) \qquad 23.$$

such that

$$\boldsymbol{y}_p^*\left(y_{-p}, X_p, \varepsilon_p\right) = s_p^j \iff \varepsilon_p^*\left(s_p^j, y_{-p}, X_p\right) < \varepsilon_p \leq \varepsilon_p^*\left(s_p^{j+1}, y_{-p}, X_p\right). \qquad 24.$$

Fix an action profile $y \equiv (y_p)_{p=1}^P$, where $y_p = s_p^j \in \mathcal{S}_p$ for each p (again, we write j instead of j_p for notational simplicity). Let $S(y)$ be as described in Equation 19. Strict concavity and independent mixing imply that if y is played with positive probability in an NE, then the support of this NE must

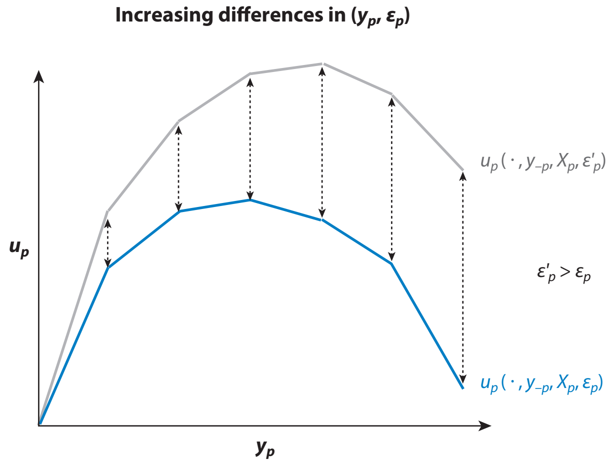

Figure 6

Increasing differences in (y_p, ε_p).

be a subset of $S(y)$. By definition, the profile y can be played in an NE only if $s_p^j = y_p^*(a_{-p}, X_p, \varepsilon_p)$ for some $a_{-p} \in S(y)$ and this is true for all p. Let $X \equiv (X_p)_{p=1}^P$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_P)'$. We obtain

$$\begin{aligned} \mathcal{R}_p(y, X_p) &= \bigcup_{a_{-p} \in S(y)} \left[\varepsilon_p^*\left(s_p^j, a_{-p}, X_p\right), \varepsilon_p^*\left(s_p^{j+1}, a_{-p}, X_p\right)\right], \\ \mathcal{R}(y, X) &= \mathcal{R}_1(y, X_1) \times \cdots \times \mathcal{R}_P(y, X_P). \end{aligned} \tag{25}$$

Let the NE outcomes $\mathcal{E}$ be as described in Definition 8. Pick any action profile y. If we maintain the assumption that the observed choice profile Y is an NE outcome, then

$$\mathbb{1}\{Y = y\} \leq \mathbb{1}\{y \in \mathcal{E}\} \leq \mathbb{1}\{\varepsilon \in \mathcal{R}(y, X)\}.$$

In particular, we obtain

$$\mathbb{1}\{\mathcal{R}(Y, X) \in \mathcal{C}\} \leq \mathbb{1}\{\varepsilon \in \mathcal{C}\} \quad \forall\, \mathcal{C} \subset \mathbb{R}^P.$$

And since X is observable to the econometrician, inference can ultimately be based on

$$Pr\left(\mathcal{R}(Y, X) \subseteq \mathcal{C} | X\right) \leq Pr\left(\varepsilon \in \mathcal{C} | X\right) \quad \forall\, \mathcal{C} \subset \mathbb{R}^P, \text{a.e. } X. \tag{26}$$

If we limit attention to PSNE and assume that Y is always the realization of a PSNE, the sets in Equation 25 would simply become

$$\mathcal{R}_p(y, X_p) = \left[\varepsilon_p^*\left(s_p^j, y_{-p}, X_p\right), \varepsilon_p^*\left(s_p^{j+1}, y_{-p}, X_p\right)\right].$$

5.1.2.1. A parametric model. Suppose we parametrize payoff functions as $u_p(y_p, y_{-p}, X_p, \varepsilon_p | \gamma_p)$. Denote $\gamma \equiv (\gamma_p)_{p=1}^P$. We now have parametric expressions for the thresholds described in

Equation 23. Denote them by $\varepsilon_p^*(s_p^j, y_{-p}, X_p|\gamma_p)$ and let

$$\mathcal{R}_p(y, X_p|\gamma_p) = \bigcup_{a_{-p}\in S(y)} \left[\varepsilon_p^*\left(s_p^j, a_{-p}, X_p|\gamma_p\right), \varepsilon_p^*\left(s_p^{j+1}, a_{-p}, X_p|\gamma_p\right)\right],$$

$$\mathcal{R}(y, X|\gamma) = \mathcal{R}_1(y, X_1|\gamma_1) \times \ldots \times \mathcal{R}_P(y, X_P|\gamma_P).$$

Next, suppose the joint distribution of $\varepsilon|X$ is also parametrized. For simplicity, suppose this parametrization assumed $\varepsilon \perp X$ and $\varepsilon \sim G(\cdot|\rho)$. The parameters of the model are $\theta \equiv (\gamma, \rho) \in \Theta$ (Θ is the parameter space). For a given set $\mathcal{C}$, let

$$H(\mathcal{C}, X, \theta) = Pr\,(\mathcal{R}(Y, X|\gamma) \subseteq \mathcal{C}|X) - Pr(\varepsilon \in \mathcal{C}|\rho).$$

Let $\overline{\mathcal{C}}$ denote a prespecified class of sets in $\mathbb{R}^P$. Based on Equation 26, the identified set for θ can be described as

$$\Theta_I = \left\{\theta \in \Theta \colon H\,(\mathcal{C}|X, \theta) \leq 0 \quad \text{a.e}\ X, \forall\, \mathcal{C} \in \overline{\mathcal{C}}\right\}. \qquad 27.$$

5.1.2.2. Example: a two-player game. As an illustration, Aradillas-López & Rosen (2019) focus on a two-player game where payoff functions are modeled as

$$u_p(y_p, y_{-p}, X_p, \varepsilon_p|\gamma_p) = y_p \times \left(\delta + X_p'\beta - \Delta_p \cdot y_{-p} - \eta \cdot y_p + \varepsilon_p\right),$$

with $\gamma_p \equiv (\delta, \beta', \Delta_p, \eta)'$. Concavity will be satisfied by specifying $\eta > 0$ in the parameter space. The increasing-differences property is satisfied by this parametrization for any γ_p. Aradillas-López & Rosen (2019) focus, for illustration purposes, on a strategic-substitutes case. Therefore, $\Delta_p \geq 0$ in the parameter space for $p = 1, 2$. The action space in this example is assumed to be $\mathcal{S}_p = \{0, 1, 2, \ldots\}$. Given this specification, the thresholds in Equation 23 have the functional form

$$\varepsilon_p^*(y_p, y_{-p}, X_p|\gamma_p) = \begin{cases} -\infty & \text{if } y_p = 0, \\ \eta \cdot (2y_p - 1) + \Delta_p y_{-p} - \delta - X_p'\beta & \text{if } y_p \geq 1. \end{cases}$$

The joint distribution of $\varepsilon = (\varepsilon_1, \varepsilon_2)'$ is parameterized as a Farlie-Gumbel-Morgenstern (FGM) copula $G(\varepsilon|\rho)$ with logistic marginal distributions.

5.1.2.3. Identification results assuming pure-strategy Nash equilibrium. Aradillas-López & Rosen (2019) assume that players always choose a PSNE.[11] As discussed above, this simplifies the set $\mathcal{R}(y|X, \gamma)$ to

$$\mathcal{R}(y|X, \gamma) = \mathop{\times}_{p=1}^{2} \left[\varepsilon_p^*(y_p, y_{-p}, X_p|\gamma_p), \varepsilon_p^*(y_p + 1, y_{-p}, X_p|\gamma_p)\right].$$

Ruling out mixed strategies in the binary-choice version of this game produces a unique prediction for $Y_1 + Y_2$ for all coexisting equilibria [see Section 4.1.1.3 and Berry & Tamer (2007, section 2.4)], and all payoff parameters can be identified from there. However, this is no longer the case as soon as we introduce more than two possible actions. With our shape restrictions, ruling out mixed

[11]Existence of a PSNE w.p.1 in this case follows from Tarski's fixed point theorem [see, e.g., Topkis (1998, section 2.5) and Vives (1999, theorem 2.2)].

strategies only pins down a parametric expression for $Pr(Y = (0,0)|X)$, because, if (0, 0) is a PSNE, it is the only PSNE. Let $\lambda \equiv \eta - \delta$. Then, we find that

$$Pr(Y = (0,0)|X) = G(\lambda - X_1'\beta, \lambda - X_2'\beta|\ \rho).$$

Let $\overline{\theta} \equiv (\lambda, \beta', \rho)'$. Then, $\overline{\theta}$ can be point-identified and estimated using, for example, maximum likelihood estimation (MLE).

5.1.2.4. Inference. Let $\theta^* \equiv (\eta, \Delta_1, \Delta_2)$, and note that $\theta = (\overline{\theta}, \theta^*)$. Aradillas-López & Rosen (2019) propose an inferential procedure where $\overline{\theta}$ is first estimated using MLE and then inference for the remaining subset of parameters θ^* is conducted using the information from the conditional moment inequalities described in Equation 27 for a prespecified class[12] of sets $\overline{\mathcal{C}}$. An outline of their procedure is as follows. Let f_x denote the density of X. For a given X and θ, let

$$\begin{aligned}
T(\mathcal{C}, X, \theta) &= (Pr(\mathcal{R}(Y, X)|\gamma) \subseteq \mathcal{C}|X) - Pr(\varepsilon \in \mathcal{C}|\rho)) \cdot f_x(X),\\
\overline{m}(\mathcal{C}, \theta) &= E_X[\max\{T(\mathcal{C}, X, \theta), 0\}], \text{ and}\\
\overline{m}(\theta) &= \sum_{\mathcal{C} \in \overline{\mathcal{C}}} \overline{m}(\mathcal{C}, \theta).
\end{aligned}$$

Note that $\overline{m}(\theta) \geq 0$ for all θ and $\overline{m}(\theta) = 0$ if and only if $T(\mathcal{C}, X, \theta) \leq 0$ a.e. X, for all $\mathcal{C} \in \overline{\mathcal{C}}$. Density weighting has computational and theoretical advantages. The identified set in Equation 27 can be rewritten as

$$\Theta_I = \left\{\theta \in \Theta : \overline{m}(\theta) = 0\right\}.$$

Suppose for illustration purposes that all the elements in $X \in \mathbb{R}^d$ are continuously distributed. Aradillas-López & Rosen (2019) propose an estimator for $\overline{m}(\theta)$ of the following form:[13]

$$\begin{aligned}
\widehat{T}(\mathcal{C}, x, \theta) &= \frac{1}{n \cdot h_n^d} \sum_{i=1}^{n} (\mathbb{1}\{\mathcal{R}(Y_i, x)|\gamma) \subseteq \mathcal{C}\} - Pr(\varepsilon \in \mathcal{C}|\theta)) K\left(\frac{X_i - x}{h_n}\right),\\
\widehat{\overline{m}}(\mathcal{C}, \theta) &= \frac{1}{n} \sum_{i=1}^{n} \widehat{H}(\mathcal{C}|X_i, \theta) \cdot \mathbb{1}\{\widehat{H}(\mathcal{C}|X_i, \theta) \geq -b_n\}, \text{ and}\\
\widehat{\overline{m}}(\theta) &= \sum_{\mathcal{C} \in \overline{\mathcal{C}}} \widehat{\overline{m}}(\mathcal{C}, \theta),
\end{aligned}$$

where $b_n \to 0$ is a nonnegative sequence converging to zero at an appropriate rate. Let

$$\Theta_I^* = \left\{\theta \in \Theta_I : H(\mathcal{C}|X, \theta) < 0 \ \text{ a.e. } \ X, \forall\, \mathcal{C} \in \overline{\mathcal{C}}\right\}.$$

This is the set of all parameter values that satisfy Equation 27 as strict inequalities almost surely. Aradillas-López & Rosen (2019) describe conditions (smoothness and regularity conditions of conditional moments, manageability and empirical-process conditions for the functional forms

[12] Using the results obtained by Chesher & Rosen (2017), Aradillas-López & Rosen (2019) characterize the class of core-determining sets $\mathcal{C}^*$ that leads to a sharp characterization of the identified set in Equation 27. In their empirical application, for computational simplicity they use a subset of this class.
[13] Notice that density weighting in the definition of $T(\mathcal{C}, X, \theta)$ helps produce a simple estimator.

assumed for payoffs and for the parametric distribution of ε, as well as kernel conditions and bandwidth convergence restrictions) such that

$$\sqrt{n}\cdot\widehat{\overline{m}}(\theta):\begin{cases}\xrightarrow{p} 0 \quad \forall\,\theta\in\Theta_I^*,\\ \xrightarrow{d} \mathcal{N}\left(0, E\left[\varphi_n(Y_i,X_i,\theta)^2\right]\right) \quad \forall\,\theta\in\Theta_I\setminus\Theta_I^*,\\ \xrightarrow{p} +\infty \quad \forall\,\theta\notin\Theta_I.\end{cases}$$

Let $\widehat{\overline{\theta}}_{\text{MLE}}$ be the MLE estimator of $\overline{\theta}$, and, for a given $\theta=(\overline{\theta},\theta^*)$, let

$$\widehat{V}(\theta)=\sqrt{n}\cdot\begin{pmatrix}\widehat{\overline{\theta}}_{\text{MLE}}-\overline{\theta}\\ \widehat{\overline{m}}(\theta).\end{pmatrix}.$$

The proposal advanced by Aradillas-López & Rosen (2019) is to construct a Wald-type statistic based on $\widehat{V}(\theta)$. However, this will require some form of regularization since the asymptotic variance of $\sqrt{n}\cdot\widehat{\overline{m}}(\theta)$ will be zero if $\theta\in\Theta_I^*$. Let

$$\widehat{t}(\theta)=\widehat{V}(\theta)'\widehat{\Sigma}(\widehat{\overline{\theta}}_{\text{MLE}},\theta^*)^{-1}\widehat{V}(\theta),$$

where $\widehat{\Sigma}(\widehat{\overline{\theta}}_{MLE},\theta^*)$ is a regularized estimator of the asymptotic variance of $\widehat{V}(\theta)$. Let $r\equiv\dim(\overline{\theta})$. Aradillas-López & Rosen (2019) show that

$$\widehat{t}(\theta):\begin{cases}\xrightarrow{d} \chi_r^2 & \text{if } \theta\in\Theta_I^*,\\ \xrightarrow{d} \chi_{r+1}^2 & \text{if } \theta\in\Theta_I\setminus\Theta_I^*,\\ \xrightarrow{p} +\infty & \text{if } \theta\notin\Theta_I.\end{cases}$$

These asymptotic properties immediately suggest how to construct a confidence set (CS) for θ. For a prespecified target coverage probability $1-\alpha$, let $c(\chi_d^2,1-\alpha)$ denote the $(1-\alpha)$-th quantile of the χ_d^2 distribution. By the properties of the chi-squared distribution, we have $c(\chi_{r+1}^2,1-\alpha)>c(\chi_r^2,1-\alpha)$. Let

$$\widehat{\text{CS}}_{1-\alpha}^{\theta}=\left\{\theta\in\Theta\colon \widehat{t}(\theta)\le c(\chi_{r+1}^2,1-\alpha)\right\}. \qquad 28.$$

Aradillas-López & Rosen (2019) show that $\widehat{\text{CS}}_{1-\alpha}^{\theta}$ satisfies $\lim_{n\to}\inf_{\theta\in\Theta_I} Pr(\theta\in\widehat{\text{CS}}_{1-\alpha}^{\theta})\ge 1-\alpha$. This inferential approach has the computational advantage that critical values do not have to be obtained by resampling methods due to the asymptotically pivotal properties of the statistic used in its construction.

5.2. Inference in Games with Incomplete Information

Aradillas-López & Gandhi (2016) study ordered discrete games with incomplete information under the assumption of BNE behavior. Their first assumption is that payoff functions can be expressed as $u_p(y_p, y_{-p}, Z_p)$ with $Z_p=(X,\varepsilon_p)$, where ε_p is privately observed by player p and X is publicly observed by all players as well as the econometrician (ε_p is not restricted to be a scalar). Their next assumption is that payoffs can be expressed as

$$u_p(Y_p,Y_{-p},Z_p)=u_p^a(y_p,Z_p)-u_p^b(y_p,Z_p)\cdot\eta_p(y_{-p},X), \qquad 29.$$

where u_p^a, u_p^b, and η_p are scalar functions. The key feature of Equation 29 is that η_p depends on Z_p solely through X. The inferential object of interest for Aradillas-López & Gandhi (2016) is the strategic index η_p. Players' private information is assumed to be mutually independent conditional on X, and beliefs are assumed to be conditioned on X. For any set of beliefs $\pi_{-p} : \mathcal{S}_{-p} \to [0,1]$, the expected utility of player p of choosing $Y_p = y_p$ is given by[14]

$$\begin{aligned}\overline{u}_p(y_p, \pi_{-p}, Z_p) &= \sum_{y_{-p} \in \mathcal{S}_{-p}} \pi_{-p}(y_{-p}|X) \cdot u_p(y_p, y_{-p}, Z_p) \\ &= u_p^a(y_p, Z_p) - u_p^b(y_p, Z_p) \cdot \overline{\eta}_p(\pi_{-p}, X)\end{aligned}$$

where $\overline{\eta}_p(\pi_{-p}, X) = \sum_{y_{-p} \in \mathcal{S}_{-p}} \pi_{-p}(y_{-p}|X) \cdot \eta_p(y_{-p}, X)$. The following shape restriction normalizes the strategic meaning of the index η_p.

Condition 3 (marginal benefit of Y_p is nonincreasing in η_p). For a.e. Z_p and any $y, y' \in \mathcal{S}_p$, we find that

$$y > y' \implies u_p^b(y, Z_p) \geq u_p^b(y', Z_p).$$

This implies that the index η_p and player p's optimal strategy are strategic substitutes, but it does not presuppose that the game itself is of strategic substitutes, since the index η_p can shift in many possible ways with opponents' actions. The constructive implication of the previous assumption is as follows. Take any pair of beliefs π_{-p} and π'_{-p}. Then, we obtain

$$\begin{aligned}&\Big[\overline{u}_p(y, \pi_{-p}, Z_p) - \overline{u}_p(y', \pi_{-p}, Z_p)\Big] - \Big[\overline{u}_p(y, \pi'_{-p}, Z_p) - \overline{u}_p(y', \pi'_{-p}, Z_p)\Big] \\ &= \Big[\overline{\eta}_p(\pi'_{-p}, X) - \overline{\eta}_p(\pi_{-p}, X)\Big] \cdot \Big[u_p^b(y, Z_p) - u_p^b(y', Z_p)\Big].\end{aligned}$$

It follows that, if $\overline{\eta}_p(\pi'_{-p}, X) \geq \overline{\eta}_p(\pi_{-p}, X)$, then we have

$$\overline{u}_p(y, \pi'_{-p}, Z_p) - \overline{u}_p(y', \pi'_{-p}, Z_p) \leq \overline{u}_p(y, \pi_{-p}, Z_p) - \overline{u}_p(y', \pi_{-p}, Z_p) \quad \forall\, y > y'.$$

Now, consider two sets of beliefs, π_{-p} and π'_{-p}, each having a unique optimal choice for the corresponding expected-utility functions. Denote them as $y_p^*(\pi_{-p}, Z_p)$ and $y_p^*(\pi'_{-p}, Z_p)$, respectively. From our previous assumption, for a.e. Z_p, we obtain

$$\overline{\eta}_p(\pi'_{-p}, X) > \overline{\eta}_p(\pi_{-p}, X) \implies \mathbb{1}\{y_p^*(\pi_{-p}, Z_p) \leq y_p\} \leq \mathbb{1}\{y_p^*(\pi'_{-p}, Z_p) \leq y_p\} \quad \forall\, y_p \in \mathcal{S}_p. \qquad 30.$$

To see why, note first that, by definition, $\overline{u}_p(y, \pi_{-p}, Z_p) - \overline{u}_p(y_p^*(\pi_{-p}, Z_p), \pi_{-p}, Z_p) < 0$ for all $y > y_p^*(\pi_{-p}, Z_p)$. But from our assumptions above, $\overline{\eta}_p(\pi'_{-p}, X) > \overline{\eta}_p(\pi_{-p}, X)$ implies

$$\overline{u}_p(y, \pi'_{-p}, Z_p) - \overline{u}_p(y_p^*(\pi_{-p}, Z_p), \pi'_{-p}, Z_p) \leq \overline{u}_p(y, \pi_{-p}, Z_p) - \overline{u}_p(y_p^*(\pi_{-p}, Z_p), \pi_{-p}, Z_p) < 0$$

for all $y > y_p^*(\pi_{-p}, Z_p)$. Thus, we must have $y_p^*(\pi'_{-p}, Z_p) \leq y_p^*(\pi_{-p}, Z_p)$, and the inequality in Equation 30 follows. The inequality in Equation 30 produces a testable implication for the model if we assume that players' beliefs are such that they always produce a unique optimal choice. Aradillas-López & Gandhi (2016) show that under these assumptions,

$$Cov(\mathbb{1}\{Y_p \leq y_p\}, \eta_p(Y_{-p}, X)|X) \geq 0 \quad \forall\, y_p \in \mathcal{S}_p, \ \text{a.e.}\, X. \qquad 31.$$

[14]The independent-mixing nature of Nash equilibrium is once again key here. Without this property, beliefs could be conditioned on the potential choice y_p, and expected payoffs would not have the structure exploited to obtain the results that follow.

The binary-choice version of this result was obtained by de Paula & Tang (2012) under assumptions of symmetry (a condition not required in Aradillas-López & Gandhi 2016). Note that if the selection mechanism $\mathcal{M}$ is degenerate w.p.1 (e.g., if there is always a unique BNE), then $Cov(\mathbb{1}\{Y_p \leq y_p\}, \eta_p(Y_{-p}, X)|X) = 0$ w.p.1. Aradillas-López & Gandhi (2016) detail inferential and testing procedures based on Equation 31 and test statistics of the type used by Aradillas-López & Rosen (2019) and described previously. They show how to test a particular functional form for the index η_p or parametrize it and construct a CS for the parameters.

6. CONCLUDING REMARKS AND DIRECTIONS FOR FUTURE RESEARCH

Recent advancements in the econometrics of partially identified models have made it possible to do robust inference in static games with multiple solutions. However, more work needs to be done to make these models a better approximation to real-world problems. Some lines of research that need to be explored further include the following.

1. Nonequilibrium models: The assumption that economic agents have correct beliefs and perfect models about others is an elegant theoretical framework, but perhaps not a realistic approximation to real-world behavior in many instances. More work needs to be done to develop econometric models that allow for incorrect beliefs and, importantly, methods that can explore more precisely how agents choose from within a space of allowable (e.g., rationalizable) beliefs. Characterizing testable implications[15] (preferably nonparametric) of different solution concepts to help researchers discriminate among competing behavioral models can produce valuable contributions to the literature.
2. Allowing a more flexible information structure in incomplete-information games: More work needs to be done to relax the assumption of conditional independence of private information. The current state of the art either assumes that the source of correlation is very simple (e.g., a commonly observed shock with finite support) or imposes a fully parametrized model. Endogeneity (i.e., correlation between observed and unobserved payoff shifters) must be allowed if these models are to be of any practical use. As in the previous point, developing nonparametric specification tests to help choose among different information structures remains an important topic.
3. Exploring richer action spaces: As we illustrated in this review, recent advances have been made to extend binary-choice games into models with richer strategy spaces. However, more work needs to be done to bring these methods closer to real-world problems—for example, by introducing multi-valued actions [e.g., adding an element of strategic interaction to the single-agent models studied by Heckman (1978)] or action spaces with unobserved, random censoring (e.g., capacity constraints).
4. Counterfactual analysis: As we discussed, many of these models are only partially identified. This can discourage practitioners from using robust inferential methods that only produce set predictions. The question then is how to achieve a balance between robustness and predictions from counterfactual analysis. One way is to develop methods to refine the set predictions for these outcomes (borrowing perhaps from recent advances in methods to do inference in a subset of parameters in partially identified models) (Kaido et al. 2019). Alternatively, we can reframe the problem entirely and do inference on a population objective

[15] The first fully nonparametric, nonexperimental test for different levels of rationality in game-theoretic models was developed by Kosenkova (2019) for first-price auctions.

function based on the counterfactual policy we want to analyze. Results from the theoretical literature on policy robustness (see Hansen & Sargent 2016) can be explored.

Most of the methods described in Sections 4 and 5 have the computational advantage of not requiring a search for all existing solutions. In fact, many do not require searching for any solution but rely instead on necessary conditions for an outcome to be a solution. This stands in contrast, for example, to dynamic models (games and single-agent models), which typically rely on computing a fixed point and devote significant efforts to computational methods that can find solutions (Aguirregabiria & Mira 2007, Bajari et al. 2007) with good properties. Instead, the most important computational challenges in most of the models studied here are the same ones encountered in partially identified models. These include developing computationally feasible methods to perform inference for the sharp identified set (Beresteanu & Molinari 2008, Beresteanu et al. 2011, Galichon & Henry 2011, Chesher & Rosen 2017), improving on existing grid search procedures in the estimation of confidence sets (Kline & Tamer 2016), and constructing nonconservative confidence regions for a subset of parameters or lower-dimensional functionals of the parameters in the model (Kaido et al. 2019, Torgovitsky 2019). Inference in static games will continue to benefit from research in these areas.

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