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The Economics of Insurance: A Derivatives-Based Approach

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Abstract

This article revisits the economics of insurance using insights from derivatives pricing and hedging. Applying this perspective, I emphasize the following insights applicable to insurance. First, I provide a valid justification for the use of arbitrage-free insurance premiums. This justification applies in both complete and incomplete markets. Second, I demonstrate the importance of diversifiable idiosyncratic risk for the determination of insurance premiums. And third, analyzing the insurance industry using the functional approach, I show the importance of derivatives and the synthetic construction of derivatives for reducing an insurance company's insolvency risk.

1. INTRODUCTION

This article integrates two related but distinct literatures: the economics of insurance and the pricing and hedging of derivative securities. This integration is motivated by the simple fact that insurance contracts are specific types of derivatives. One can easily see this by recalling the definitions of insurance and derivatives.

> **Insurance:** coverage by contract whereby one party undertakes to indemnify or guarantee another against loss by a specified contingency or peril. (Merriam-Webster Dictionary, **https://www.merriam-webster.com/dictionary/insurance**)

> **Derivative:** a contract or security that derives its value from that of an underlying asset (such as another security) or from the value of a rate (as of interest or currency exchange) or index of asset value (such as a stock index). (Merriam-Webster Dictionary, **https://www.merriam-webster.com/dictionary/derivative**)

From these two definitions, we see that an insurance contract is a special case of a derivative security where the underlying asset in the insurance contract is the insured property and the value of the insurance contract (the derivative) is derived from the value (the loss) in the underlying property. For example, a property could be a car, home, or human capital (life).

As such, we would expect that the economics of insurance can be better understood by combining the classical insights of insurance with the more recent insights from derivatives—in particular, the arbitrage-free valuation and synthetic construction methodology as well as insights from the functional approach to financial institutions. The purpose of this article is to provide such an integrated review of the economics of insurance. For simplicity of presentation, we present the key concepts using a static model. Dynamic extensions are discussed. Specific insurance markets are used to illustrate all of the important concepts.

The key insights that financial economics, derivatives pricing, and hedging theory bring to our understanding of insurance are

- the justification for the use of arbitrage-free insurance premiums,
- the computation of arbitrage-free insurance premiums in both complete and incomplete markets,
- the importance of diversifiable idiosyncratic event loss risk for the determination of insurance premiums, and
- the synthetic construction of insurance contracts and their use to reduce insolvency risk.

An outline of this article is as follows: Section 2 reviews the arbitrage-free valuation insights from financial economics and derivatives pricing and hedging theory. Section 3 discusses the notion of diversifiable idiosyncratic risk and its importance for valuation in incomplete markets. Insurance premiums are the content of Section 4. Sections 5 and 6 present the theories that explain the demand and supply of insurance, respectively. Section 7 combines the previous two sections and discusses equilibrium and endogenous insurance contract design. All proofs are in Section 8 unless otherwise stated. Section 9 concludes.

2. THE ARBITRAGE-FREE VALUATION

This section reviews the arbitrage-free pricing methodology in the context of a static financial market model. This is the standard model used in the derivative pricing literature.

2.1. The Setup

We assume that the uncertainty in the market is characterized by a complete probability space $(\Omega, \mathcal{F}, P)$, where Ω is the state space, $\mathcal{F}$ is the event space (a σ−algebra), and P is a probability measure.[1] Complete means that $\mathcal{F}$ contains all P null sets. P represents the statistical probability measure, which is the probability generating the realized prices and losses observed in the market. Trading are underlying assets (property to be insured) and a money market account (mma). Denote the values of an asset i (property to be insured) as $v_i(t)$ at times $t = 0, 1$ for $i = 1, \ldots, n$, where $v_i(0) \geq 0$ is a constant and $v_i(1)$ is a random variable for all i.[2] The mma has a unit investment at time 0 and pays $1 + r$, with $r > 0$ a constant at time 1.

The mma and asset (and insurance) markets are assumed to be frictionless and competitive. By frictionless we mean there are no transaction costs and no trading constraints (e.g., short sale constraints or margin requirements). Competitive means that investors act as price takers with respect to asset prices, including the insurance contract premiums. A portfolio of the mma $(i = 0)$ plus assets $(i = 1, \ldots, n)$ is given by a vector $(\theta_0, \theta_1, \ldots \theta_n)$ representing the number of each held at time 0. No restrictions are imposed on the share holdings because markets are frictionless.

Definition 1 (No arbitrage). No arbitrage means that there exists no portfolio $(\theta_0, \theta_1, \ldots \theta_n)$ of the mma plus assets that has initial value

$$\sum_{i=1}^{n} \theta_i v_i(0) + \theta_0 \cdot 1 = 0,$$

with

$$P\left(\sum_{i=1}^{n} \theta_i v_i(1) + \theta_0 \cdot (1 + r) \geq 0\right) = 1$$

and

$$P\left(\sum_{i=1}^{n} \theta_i v_i(1) + \theta_0 \cdot (1 + r) > 0\right) > 0.$$

An arbitrage opportunity is a portfolio that costs zero to construct, never loses value, and with positive probability has a positive payoff.

Theorem 1 (The first fundamental theorem of asset pricing). The market satisfies no arbitrage if and only if there exists a probability measure Q equivalent[3] to P such that

$$v_i(0) = \frac{E^Q\left(v_i(1)\right)}{1 + r} \tag{1}$$

for $i = 1, \ldots, n$, where $E^Q(\cdot)$ denotes expectation under Q.

Proof. See Jarrow (2019). □

Let the set of equivalent probability measures satisfying Equation 1 be denoted $\mathcal{M}$. Alternatively stated, the first fundamental theorem says that the market satisfies no arbitrage if and only if there exists a $Q \in \mathcal{M}$.

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[1] For the definitions of a σ−algebra and probability measure, see Billingsley (1986).

[2] A random variable is a mapping from Ω into the real line, which is $\mathcal{F}$-measurable with respect to the Borel σ−algebra on the real line [for the definitions of measurability and the Borel σ−algebra, see Billingsley (1986)].

[3] Equivalent means that Q and P agree on zero probability events in $\mathcal{F}$.

Definition 2 (Complete market with respect to $Q \in \mathcal{M}$). Assume the market satisfies no arbitrage. The market is complete with respect to $Q \in \mathcal{M}$ if for any random variable $X \geq 0$ (a derivative) realized at time 1 with $E^Q(X) < \infty$ there exists a portfolio $(\theta_0, \theta_1, \ldots \theta_n)$ with initial value

$$x = \sum_{i=1}^{n} \theta_i v_i(0) + \theta_0 \cdot 1$$

such that

$$\sum_{i=1}^{n} \theta_i v_i(1) + \theta_0 \cdot (1 + r) = X.$$

This definition states that in an arbitrage-free and complete market, given any derivative's random payoff $X \geq 0$ at time 1, there exists a portfolio of the traded assets that replicates the payoff to the derivative. This portfolio is called the synthetic derivative, and the payoff is said to be attained by synthetic construction. Synthetic construction is the key insight underlying risk-neutral valuation, discussed in Theorem 3 below. In this definition, we say that the portfolio $(\theta_0, \theta_1, \ldots \theta_n)$ generates the payoff $X \geq 0$.

Theorem 2 (Second fundamental theorem of asset pricing). Given no arbitrage (i.e., there exists a $Q \in \mathcal{M}$), the market is complete with respect to Q if and only if Q is unique.

Proof. See Jarrow (2019). □

2.2. Complete Markets

We get the following theorem and corollary in an arbitrage-free and complete market.

Theorem 3 (Risk-neutral valuation). Assume no arbitrage and that the market is complete with respect to $Q \in \mathcal{M}$.
Given any random variable (derivative) $X \geq 0$ with $E^Q(X) < \infty$, then

$$x = \frac{E^Q(X)}{1 + r}, \qquad 2.$$

where $x = \sum_{i=1}^{n} \theta_i v_i(0) + \theta_0 \cdot 1$ and $(\theta_0, \theta_1, \ldots \theta_n)$ is the portfolio generating X.

This theorem follows directly from the first and second fundamental theorems of asset pricing. It follows because in a complete market, given a derivative's random payoff $X \geq 0$, there exists a portfolio that synthetically constructs the same payoffs. And we know the cost of constructing this portfolio, which is x. Then, no arbitrage implies that the arbitrage-free price of the derivative must be equal to its cost of construction x.

Note also that Equation 2 can be interpreted as a present value formula; i.e., to get the present value of a future random cash flow $X \geq 0$, take its expected value under Q and discount using the risk-free rate r. Because the future cash flow need not be riskless, the risk adjustment necessary to determine the present value is included in the probability Q. Alternatively stated, the difference between the probabilities P and Q is that Q contains an adjustment for risk, so that $E^Q(X)$ can be interpreted as the certainty equivalent of the random payoff $X \geq 0$.

Since Equation 2 is the present value formula with no adjustment for risk in a hypothetical risk-neutral economy, the probability Q is often called a risk-neutral probability measure.[4]

[4]In this hypothetical economy, all traders are risk-neutral and their beliefs equal Q.

Corollary 1 (Systematic risk return relation). Assume no arbitrage and that the market is complete with respect to $Q \in \mathcal{M}$.
Assume $x > 0$ and $E^P(X) < \infty$.
Let $R = \frac{X-x}{x}$ denote the underlying asset's or derivative's return.
Then,

$$E^P(R) = r - cov^P\left(R, \frac{dQ}{dP}\right), \quad 3.$$

where $E^P(\cdot)$ denotes expectation under P and $\frac{dQ}{dP}$ is the Radon-Nykodym derivative.[5]

This corollary gives the risk-return relation for any asset or derivative. In Equation 3, the quantity $\frac{dQ}{dP}$ corresponds to the market's state price density. We see that the expected return of an asset or derivative is compensated for by systematic risk, as represented by the covariance between the asset's return and the state price density. This relation gives another perspective on arbitrage-free pricing. It shows that in a complete market, any expected return from an asset or derivative in an arbitrage-free setting contains a risk premium adjustment for systematic risk. Equilibrium pricing is not needed for this risk-return relation, because the risk-neutral probability is unique.

Remark 1 (Arbitrage-free portfolio valuation in an incomplete market). Even in an incomplete market, Equations 2 and 3 hold for any portfolio $(\theta_0, \theta_1, \ldots \theta_n)$ under the assumption of no arbitrage, i.e.,

$$x = \frac{E^Q(X)}{1+r}$$

and

$$E^P(R) = r - cov^P\left(R, \frac{dQ}{dP}\right),$$

where $X = \sum_{i=1}^{n} \theta_i v_i(1) + \theta_0 \cdot (1+r)$. This follows because the portfolio is a linear combination of the underlying assets. And, even though derivatives whose payoffs cannot be replicated may exist, linear combinations are obtainable as simple buy-and-hold trading strategies.

In an incomplete market, in contrast to a complete market, there are an infinite number of probabilities $Q \in \mathcal{M}$ for which these formulas are valid. Nonetheless, all risk-neutral probabilities $Q \in \mathcal{M}$ give the same initial value, $x = \sum_{i=1}^{n} \theta_i v_i(0) + \theta_0 \cdot 1$, for creating the portfolio. The difficulty arises in an incomplete market when considering a derivative X that is not a linear combination of the underlying assets and for which no replicating portfolio exists. In this case, the different $Q \in \mathcal{M}$ give different values for the derivative, hence, different risk-return relations.

2.3. Incomplete Markets

In an arbitrage-free but incomplete market, we obtain the following theorem.

Theorem 4 (Arbitrage-free valuation in an incomplete market). Assume no arbitrage.
Given any random variable (derivative) $X \geq 0$ with $E^Q(X) < \infty$ for all $Q \in \mathcal{M}$.
Let x denote the arbitrage-free price. Then,

$$x \in \left[\inf_{Q \in \mathcal{M}} \left\{\frac{E^Q(X)}{1+r}\right\}, \sup_{Q \in \mathcal{M}} \left\{\frac{E^Q(X)}{1+r}\right\}\right].$$

Proof. See the work by Dana & Jeanblanc (1998, chapter 1). □

[5]For the definition of the Radon-Nykodym derivative, see Billingsley (1986).

The theorem gives a range of arbitrage-free values for a derivative when its random payoff cannot be replicated via trading in the underlying assets. The proof of the theorem is based on showing that

$$\sup_{Q\in\mathcal{M}}\left\{\frac{E^Q(X)}{1+r}\right\} = \inf\Bigg\{x \geq 0 : \exists\ (\theta_0, \theta_1, \ldots \theta_n)\, s.t.$$

$$x = \sum_{i=1}^{n}\theta_i v_i(0) + \theta_0 \cdot 1,\ \sum_{i=1}^{n}\theta_i v_i(1) + \theta_0 \cdot (1+r) \geq X\Bigg\}$$

and

$$\inf_{Q\in\mathcal{M}}\left\{\frac{E^Q(X)}{1+r}\right\} = \sup\Bigg\{x \geq 0 : \exists\ (\theta_0 = -x, \theta_1, \ldots \theta_n)\, s.t.$$

$$0 = \sum_{i=1}^{n}\theta_i v_i(0) - x,\ \sum_{i=1}^{n}\theta_i v_i(1) - x(1+r) + X \geq 0\Bigg\}.$$

This implies that the arbitrage-free prices must lie within the range generated by the cost of superreplicating the derivative's payoffs and subreplicating them. The cost of superreplication is the smallest investment x needed to generate a portfolio in the underlying assets whose time 1 payoffs exceed the derivative's payoffs with P probability one. The cost of subreplication is the largest borrowing that, when invested in the underlying assets, the time 1 value of this portfolio is covered by the derivative's payoffs with P probability one.

Alternatively, this theorem states that given any risk-neutral probability measure $Q \in \mathcal{M}$, the quantity $\frac{E^Q(X)}{1+r}$ represents an arbitrage-free value. We see here that there are, in general, an infinite number of such arbitrage-free prices for the derivative $X \geq 0$. To determine a unique price for the derivative that cannot be replicated, a different approach needs to be employed. Four alternatives are commonly used.

One alternative is if the traded derivatives, in conjunction with the underlying assets, together complete the market. In this case, again, the risk-neutral probability measure is unique. The second alternative is if the risk of the underlying derivative is diversifiable in a large portfolio, in which case the derivative's return reflects no systematic risk (see Section 3). The third alternative is to consider an investor's optimal portfolio problem to identify how they value future random cash flows, i.e., to identify the individual's risk-neutral probability measure Q. This is called indifference pricing. Last, the fourth alternative is to use the risk-neutral probability measure implied by an equilibrium (for more details on these four alternative approaches, see Jarrow 2019).

Therefore, in an incomplete market we get the following corollary.

Corollary 2 (Systematic risk-return relation). Assume no arbitrage and that $x > 0$ and $E^P(X) < \infty$.
Let $R = \frac{X-x}{x}$ denote the underlying asset's or derivative's return.
Fix any $Q \in \mathcal{M}$. Then,

$$E^P(R) = r - cov^P\left(R, \frac{dQ}{dP}\right) \qquad 4.$$

is a possible risk-return relation.

As before, this corollary shows that any asset's expected return in an arbitrage-free setting contains a risk-premium adjustment for systematic risk. To use this systematic risk-return relation, in general, we need a rule to identify a unique $Q \in \mathcal{M}$. As discussed above, there are three exceptions. First, due to Remark 1, for the underlying assets themselves, uniqueness of the risk-neutral probability is irrelevant because all risk-neutral probabilities give the same compensation for systematic risk. Second, if in conjunction with the underlying assets the derivatives themselves complete the market, uniqueness of the risk-neutral probability results and no rule is needed. Third, if the derivative's risk is diversifiable, it exhibits no systematic risk and the risk-neutral probability is irrelevant to the derivative's risk-return relation. We discuss this in the next section. Indifference or equilibrium pricing is needed only when none of the special cases apply.

3. DIVERSIFIABLE RISK

As noted above, there are special cases within an incomplete market where exact pricing of a derivative results. These cases occur when, due to diversification, no risk premium is associated with the risk of the asset's price movements because the risk can be diversified away in a large portfolio. This means, of course, that in a large portfolio the randomness in the assets' payoffs (some above and some below the mean) acts as to cancel each other out so that the portfolio's payoff approaches its expected value. This insight is due to work by Merton (1976) and Ross (1976). Consider the market introduced in Section 2, and assume that this market is arbitrage-free, i.e., that there exists a $Q \in \mathcal{M}$.

3.1. Independent and Identically Distributed Risks

Assume that the random asset prices $v_i(1)$ for $i = 1, \ldots, n$ are independent and identically distributed (i.i.d.) risks and uniformly bounded, i.e., $|v_i(1)| \leq K$ for a constant $K > 0$.[6] Denote the common expected time 1 value as $E^P(v_i(1)) = \mu_v^P > 0$ under P and $E^Q(v_i(1)) = \mu_v^Q > 0$ under Q.

For i.i.d. assets, we get the following theorem.

Theorem 5 (Risk-neutral value equals discounted actuarial expected value).

$$v_i(0) = \frac{\mu_v^P}{1+r}, \tag{5}$$

i.e.,

$$\mu_v^Q = \mu_v^P. \tag{6}$$

And,

$$E^P(R_v) = r,$$

where $R_v = \frac{v_i(1) - v_i(0)}{v_i(0)}$.

This theorem states that when an asset's risk is diversifiable in a large portfolio, the risk is nonsystematic and earns no risk premium. This result holds in both a complete and an incomplete market.

3.2. A Factor Model

Assume that the random asset prices satisfy the factor model $v_i(1) = Z + \eta_i$, where η_i are i.i.d. and uniformly bounded, i.e., $|\eta_i| \leq K$ for a constant $K > 0$. Denote $E^P(\eta_i) = \mu_\eta^P$ under P and

[6]Any equalities and inequalities for random variables hold with P probability one.

$E^Q(\eta_i) = \mu_\eta^Q$ under Q. Let the random variable Z be such that there exists an integer $k > 0$ where for all $n \geq k$ there exists a portfolio $(\theta_0, \theta_1, \ldots \theta_k)$ with initial value

$$z = \sum_{i=1}^{k} \theta_i v_i(0) + \theta_0 \cdot 1,$$

such that

$$\sum_{i=1}^{k} \theta_i v_i(1) + \theta_0 \cdot (1+r) = Z.$$

This means that this payoff Z can be synthetically constructed using only k of the existing assets. Because Z can be synthetically constructed, any $Q \in \mathcal{M}$ will give the same initial cost of construction (present value) $z = \frac{E^Q(Z)}{1+r}$. We can now prove the following theorem.

Theorem 6 (Risk-neutral valuation in a factor model).

$$v_i(0) = z + \frac{\mu_\eta^P}{1+r}, \qquad 7.$$

i.e., for any $Q \in \mathcal{M}$,

$$\mu_\eta^Q = \mu_\eta^P. \qquad 8.$$

And,

$$E^P(R_v) = r - cov^P\left(Z, \frac{dQ}{dP}\right),$$

where $R_v = \frac{v_i(1) - v_i(0)}{v_i(0)}$ and $z = \frac{E^Q(Z)}{1+r}$.

This theorem generalizes the i.i.d. case to one where the asset's risk is partly systematic and partly idiosyncratic. In this case, the systematic part (Z) earns a risk premium, but the idiosyncratic part (η_i) does not.

Remark 2 (Dynamic extensions). All of the previous theorems can be extended to a dynamic and continuous time setting (see Jarrow 2019).

4. INSURANCE PREMIUM DETERMINATION

This section uses the arbitrage-free valuation methodology to determine arbitrage-free insurance premiums. We assume that the asset (and insurance) markets are frictionless and competitive. By frictionless, we mean there are no transaction costs and no trading constraints (e.g., short sale constraints or margin requirements). Competitive means that investors act as price takers with respect to asset prices, including the insurance contract premiums.

In a complete market, if the insurer set a different and larger premium, then arbitrageurs could enter the market and provide insurance contracts at the cheaper premium, hedging them synthetically, to make arbitrage profits. In an incomplete market, if one cannot synthetically create the short position in the insurance contract to hedge the long position, then the alternative provider could still provide the insurance more cheaply by lowering the premium and still obtain proper compensation for the risks involved. Competition in the market for insurance providers should imply that the premium in an incomplete market is again equal to the arbitrage-free value. Using the arbitrage-free valuation methodology is an important insight currently not used for premium determination in the insurance industry (see Feng 2018, p. 163).

Consider a generic insurance contract on the value of a property (auto, home, human capital, mortgage, bond) with values $v_i(t)$ at times $t = 0, 1$ for $i = 1, \ldots, n$, where $v_i(0) \geq 0$ is a constant and $v_i(1)$ is a random variable. For most applications, these properties do not trade in liquid markets, implying the market is incomplete. Both complete and incomplete market arbitrage-free pricing is discussed below. We assume that the property values are identically distributed.

The formal description of the insurance contract (the derivative) is as follows. At time 0, the insured pays a premium $p_i > 0$. This premium guarantees that if there is a loss on the property, which is realized at time 1, the insurer will cover the loss. The random loss on the property value is

$$\varepsilon_i = -min[v_i(1) - v_i(0), 0] \geq 0.$$

Hence, the insurance contract pays off $\varepsilon_i \geq 0$ at time 1. Because the property values are identically distributed, denote $E^P(\varepsilon_i) = \mu_\varepsilon^P > 0$ as the actuarial expected loss on the ith property for all i. Note that the actuarial expected loss does not depend on the property i because the properties are identically distributed.[7]

By construction, the time 0 value of this insurance contract is zero. This guarantees that at the time the insurance contract is written, no cash exchanges hands except for the payment of the initial premium.

Let us decompose the insurance premium received per product into three components:[8]

$$p_i = \frac{E^P(\varepsilon_i)}{1+r} + \pi_i + c_i, \tag{9}$$

where

- $\frac{E^P(\varepsilon_i)}{1+r} = \frac{\mu_\varepsilon^P}{1+r} > 0$ is the present value of the actuarial expected loss on the property,
- $\pi_i \geq 0$ is the risk-premium adjustment per contract at time 0, and
- $c_i \geq 0$ is the cost of creating and servicing the contract (allocated fixed and marginal cost), assumed incurred at time 0.

The random cash flows to the insurer from holding the insurance contract at time 1 are

$$I_i = \left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i + c_i\right)(1+r) - c_i(1+r) - \varepsilon_i. \tag{10}$$

This expression corresponds to the time 1 value of the insurance premium from contract i less the future value of the costs of creating and servicing the contract and any losses incurred. The only randomness here is ε_i. Hence, this derivative is equal to a long position in the mma equal to

$$\left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i + c_i\right) - c_i = \frac{E^P(\varepsilon_i)}{1+r} + \pi_i$$

and a short position in the random payoff ε_i. Consequently, we can think of the random payoff ε_i as itself a traded derivative. We use this insight below.

[7]Although obvious when stated, it is important to emphasize that the insurance contract is valued conditioned on the insured owning the policy. This implicitly implies that the loss reflects the actions of the insured after ownership. For some insurance coverages, the existence of insurance may affect the insured's behavior, which is known as moral hazard. For additional discussion, see Section 7.

[8]This decomposition is a simplification of the decomposition contained in the work by Booth et al. (2005, p. 403).

By definition, the arbitrage-free insurance premium sets the time 0 value of the insurance contract equal to zero. Given the premium's decomposition in Equation 9, determining the arbitrage-free premium is equivalent to determining the arbitrage-free risk-premium adjustment π_i embedded in the insurance contract. This is the task to which we now turn.

4.1. Complete Markets

This section determines the arbitrage-free insurance premium in a complete market.

Theorem 7 (Insurance premiums in a complete market). Assume the market is arbitrage-free and complete with respect to $Q \in \mathcal{M}$.
Then, the arbitrage-free risk-premium adjustment is

$$\pi_i = \frac{\mu_\varepsilon^Q - \mu_\varepsilon^P}{1+r} \geq 0 \qquad 11.$$

and the arbitrage-free premium is

$$p_i = \frac{\mu_\varepsilon^Q}{1+r} + c_i > 0. \qquad 12.$$

This result is quite intuitive. In a competitive insurance market, it states that an insurance contract's premium should be its expected discounted payoff, after adjusting for risk via the use of the risk-neutral probabilities $Q \in \mathcal{M}$, plus the costs of creating the contract. No ambiguity about the risk-neutral probability Q occurs, because in a complete market it is unique. Premiums based on the actuarial expected loss alone, plus costs, would not provide adequate compensation for the systematic risk embedded in the insurance contract's losses. The insurance company's risk-premium adjustment π_i provides compensation for the systematic risk of the insured losses. This is analogous to the risk-premium component in the insurance premium as discussed by Booth et al. (2005, chapter 14).

4.2. Incomplete Markets

In an incomplete market, there is no unique arbitrage-free insurance premium. The ranges of arbitrage-free insurance premiums and risk-premium adjustments are given in the following theorem.

Theorem 8 (Insurance premiums in an incomplete market).

$$\pi_i \in \left[\inf_{Q \in \mathcal{M}} \left\{ \frac{\mu_\varepsilon^Q - \mu_\varepsilon^P}{1+r} \right\}, \sup_{Q \in \mathcal{M}} \left\{ \frac{\mu_\varepsilon^Q - \mu_\varepsilon^P}{1+r} \right\} \right]$$

$$p_i \in \left[\inf_{Q \in \mathcal{M}} \left\{ \frac{\mu_\varepsilon^Q}{1+r} + c_i \right\}, \sup_{Q \in \mathcal{M}} \left\{ \frac{\mu_\varepsilon^Q}{1+r} + c_i \right\} \right].$$

As noted in this theorem, in an incomplete market, there are a range of arbitrage-free insurance premiums consistent with no arbitrage. Any risk-neutral probability $Q \in \mathcal{M}$ gives an acceptable insurance premium. To uniquely determine the risk-neutral probability to apply in an incomplete market, as discussed in Section 2.3, there are four alternatives. One alternative is if the traded insurance derivatives, in conjunction with the underlying assets, together complete the market. In this case, the risk-neutral probability measure is unique. The second alternative is if the risk of the underlying derivative is diversifiable in a large portfolio, in which case actuarial insurance premiums apply (see Section 4.3). The third alternative is to consider an investor's optimal portfolio and consumption problem and use indifference pricing. Last, the fourth alternative is to use the risk-neutral probability measure implied by an equilibrium.

4.3. Diversifiable Risk (Incomplete Market)

For insured properties that do not trade or only trade in incomplete markets, if the loss risks are diversifiable in a large portfolio of insurance contracts, then an arbitrage-free premium can still be uniquely determined. Properties with this characteristic include autos, homes, and human capital (life), excluding catastrophic events. In the traditional insurance literature, the law of large numbers is used in the determination of the actuarial component of insurance premiums when the claims are i.i.d. This is called the equivalence principle (see Feng 2018, p. 5). This use, however, is distinct from the notion of diversifiable risk used below, although both approaches use the law of large numbers for i.i.d. losses. Assume for this section that the market for insurance is arbitrage-free; i.e., there exists a $Q \in \mathcal{M}$.

4.3.1. Independent and identically distributed losses. Assume that the random losses $\varepsilon_i \geq 0$ for $i = 1, \ldots, n$ are i.i.d. under P and uniformly bounded with $|\varepsilon_i| \leq K$ for a constant $K > 0$. Denote the common expected time 1 value as $E^P(\varepsilon_i) = \mu_\varepsilon^P > 0$ under P and $E^Q(\varepsilon_i) = \mu_\varepsilon^Q > 0$ under $Q \in \mathcal{M}$. Examples would be auto insurance, term life insurance, and reinsurance contracts that renew every year. By the law of large numbers, the average loss converges to its actuarial mean with P probability one, i.e.,

$$\frac{\sum_{i=1}^n \varepsilon_i}{n} \to \mu_\varepsilon^P.$$

We can prove the following theorem.

Theorem 9 (Insurance premiums for i.i.d. losses).

$$\pi_i = 0$$

and

$$p_i = \frac{\mu_\varepsilon^P}{1+r} + c_i > 0.$$

This theorem shows that when event losses are i.i.d., the arbitrage-free insurance premium contains no systematic risk; therefore, the arbitrage-free insurance premium equals its actuarial value plus costs. In addition, since there is no systematic risk in the insured losses, in the limit the risk-premium adjustment equals zero.

4.3.2. Factor model. Assume that the random insurance losses satisfy the factor model $\varepsilon_i = Z + \eta_i$ for $i = 1, \ldots, n$, where η_i are i.i.d. and uniformly bounded with $|\eta_i| \leq K$ for a constant $K > 0$. Denote $E^P(\eta_i) = \mu_\eta^P > 0$ under P and $E^Q(\eta_i) = \mu_\eta^Q > 0$ under Q. An example would be home insurance with common loss events (earthquakes) and idiosyncratic ones.

Let the common random variable Z be such that there exists a $k > 0$, where for all $n \geq k$ there exists a portfolio $(\theta_0, \theta_1, \ldots \theta_k)$ with initial value

$$z = \sum_{i=1}^{k} \theta_i v_i(0) + \theta_0 \cdot 1,$$

such that

$$\sum_{i=1}^{k} \theta_i v_i(1) + \theta_0 \cdot (1 + r) = Z.$$

Because the common loss Z can be synthetically constructed, for any $Q \in \mathcal{M}$, we have the same initial cost of construction $z = \frac{E^Q(Z)}{1+r}$. In addition, we have the idiosyncratic losses $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \eta_i}{n} \to \mu_\eta^P$ with P probability one. Combined, these two facts yield the following theorem.

Theorem 10 (Insurance premiums for a factor model for losses).

$$\pi_i = z - \frac{E^P(Z)}{1+r} \geq 0$$

and

$$p_i = z + \frac{\mu_\eta^P}{1+r} + c_i > 0,$$

where $z = \frac{E^Q(Z)}{1+r}$ for any $Q \in \mathcal{M}$.

This theorem shows that when event losses are composed of systematic and idiosyncratic risk ($\varepsilon_i = Z + \eta_i$), the idiosyncratic risk component (η_i) of the arbitrage-free insurance premium is computed using its actuarial value and only the systematic component (Z) contains a risk premium. The risk premium reflects just the systematic risk due to Z.

4.4. Dynamic Extensions

For some products that are reinitiated every year with premiums determined by the current year conditions, the static model is sufficient. This is typical of general insurance products, including auto and home insurance.[9] It also applies to reinsurance contracts, which typically have a yearly cycle, with premiums changing each year. In contrast, many insurance contracts have a fixed multiyear maturity and the premium is fixed at the time the insurance contract is issued. In such circumstances, a dynamic model is needed. Examples include various forms of life insurance: term life, whole life, pure endowment insurance, endowment insurance, deferred whole life, increasing whole life, decreasing term life, and equity-linked life (see Melnikov 2011, section 7.3; Feng 2018, chapter 1). Additional examples include health insurance, mortgage insurance, and credit default swaps (insurance on bonds).

All of the previous arbitrage-free pricing results for the determination of insurance premiums can be extended to a dynamic market. However, in this extension the diversification argument, in general, fails to apply because the discounted values of the event losses are no longer independent. This implies that, in general, actuarial-determined premiums without an adjustment for systematic risk will not be arbitrage-free. A generic insurance contract illustrates these assertions.

4.4.1. Generic insurance.

We consider a continuous trading model on a finite horizon $[0, T]$. The uncertainty in the model is characterized by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual hypotheses and $\mathcal{F} = \mathcal{F}_T$.[10]

[9] In the insurance literature, the term general insurance applies to products not classified as life insurance (see Booth et al. 2005, p. 369).

[10] For definitions of these various terms, see Protter (2005).

Here, P is the statistical probability measure. As before, by the statistical probability measure we mean the probability P from which historical time series data are generated.

Assume traded in the market are an mma with time $t \geq 0$ value denoted $e^{\int_0^t r_u du}$, where r_t is the default-free spot rate of interest, and default-free zero-coupon bonds that pay a certain dollar at time T with time $t \geq 0$ value denoted $B(t, T) > 0$.[11] Let $\tau \in (0, T]$ be a stopping time with respect to the filtration representing an insured loss event. Consider the dates $\{0, t_1, \ldots, t_m = T\}$. Assume traded are loss event securities with payoffs $1_{\{\tau > t_k\}}$ for $k = 1, \ldots, m$, which pay a dollar at time t_k only if the loss event does not occur before t_k. Let the time $t \geq 0$ value of the loss event securities be denoted ε_t^k for $k = 1, \ldots, m$.[12]

We assume that there exists an equivalent probability measure Q, such that

$$B(t, \mathcal{T})e^{-\int_0^t r_u du} \quad \text{and} \quad \varepsilon_t^k e^{-\int_0^t r_u du}$$

are Q martingales for all $\mathcal{T} \in [0, T]$ and $k = 1, \ldots, m$. By the fundamental theorems of asset pricing, this assumption is equivalent to assuming that the markets satisfy no arbitrage and no dominance.[13]

Consider a generic insurance policy that pays off L dollars at the time an insured event occurs. The protection buyer pays a constant premium, c dollars, at the intermediate dates $\{0, t_1, \ldots, t_m = T\}$ but only up to the event time τ. After time τ, the premium payments cease and the insurance pays off the L dollars. This generic insurance contract insures only the first occurrence of the insured event. This is standard for many insurance contracts, e.g., life insurance, credit default swaps, and reinsurance contracts.[14]

Last, we assume that the default-free spot rate r_t and the loss events $\{\tau > t_k\}$ for $k = 1, \ldots, m$ are all independent under the equivalent probability measure Q. This is a reasonable assumption for most insurance contracts (e.g., life, auto, home) whose losses have little if any impact on the macroeconomy and interest rates in particular.

Then, using risk-neutral valuation, the time 0 value of the insurance policy to the insurance company is

$$E^Q\left[\sum_{k=1}^m c1_{\{\tau > t_k\}} e^{-\int_0^{t_k} r_u du} - L1_{\{\tau \leq T\}} e^{-\int_0^\tau r_u du}\right].$$

The arbitrage-free premium c is determined such that

$$0 = E^Q\left[\sum_{k=1}^m c1_{\{\tau > t_k\}} e^{-\int_0^{t_k} r_u du} - L1_{\{\tau \leq T\}} e^{-\int_0^\tau r_u du}\right]$$

or

$$c = \frac{LE^Q\left[1_{\{\tau \leq T\}} e^{-\int_0^\tau r_u du}\right]}{\sum_{k=1}^m E^Q\left[1_{\{\tau > t_k\}} e^{-\int_0^{t_k} r_u du}\right]} = \frac{LE^Q\left[1_{\{\tau \leq T\}} e^{-\int_0^\tau r_u du}\right]}{\sum_{k=1}^m B(0, t_k) Q(\tau > t_k)}. \qquad 13.$$

The last equality is due to the independence of r_t and $\{\tau > t_k\}$ under the equivalent probability measure Q and because $E^Q\left[1_{\{\tau > t_k\}}\right] = Q(\tau > t_k)$.

This arbitrage-free insurance premium is valid in either a complete or an incomplete market. In an incomplete market, however, the determination of the equivalent martingale measure Q is needed to get a unique value (as discussed above).

[11] We assume that $\int_0^T |r_u|\, du < \infty$.

[12] All the previous stochastic processes are assumed to be semi-martingales adapted to the filtration.

[13] For the definitions of these terms and the fundamental theorems of asset pricing, see Jarrow (2019).

[14] However, for other insurance contracts, such as home or auto insurance, the contract pays off for repeated occurrences of the insured event over the contract's life. This modification is easily handled in the subsequent valuation by introducing multiple event times τ_j and multiple loss payoffs L_j and by continuing the premium payments c until the contract matures.

Next, suppose that the market is incomplete. Consider insured events indexed by $i = 1, \ldots, n$, where the loss event times τ^i are assumed to be i.i.d. Unfortunately, the diversification argument cannot be extended in the dynamic setting because the random variables $e^{-\int_0^{\tau^i} r_u du} 1_{\{\tau^i \leq T\}}$ for $i = 1, \ldots, n$ are no longer independent.[15] This lack of independence is due to the discount factors $e^{-\int_0^{\tau^i} r_u du}$ being correlated. So the law of large numbers cannot be invoked to replace the risk-neutral probabilities Q with the statistical probabilities P, and the diversification argument in an incomplete market fails (for a proof of this assertion, see Section 8).

4.4.2. Specific insurance contracts. In the literature, arbitrage-free pricing of various insurance contracts with a time-to-maturity dependence as discussed above have been studied. Examples include various life insurance contracts, equity-linked annuities, and credit default swaps (see, e.g., Lee 2003; Tanskanen & Lukkarinen 2003; Barbarin 2008; Kassberger, Kiesel & Liebmann 2008; Zaglauer & Bauer 2008; Jarrow 2009).

4.5. Estimation and Computation

Actuarials have developed many insights into estimating the distributions of loss events, which is estimation of the P distributions (see Boland 2007; Borowiak & Shapiro 2014). For the determination of arbitrage-free insurance premiums, this literature is insufficient. It really only applies, as discussed above, to i.i.d. loss events for which actuarial fair insurance premiums apply. For the remaining loss events, estimation of the risk-neutral Q distributions is necessary. Fortunately, financial economists have studied this issue almost since the inception of option pricing theory. In this regard, one can use insurance derivative prices to infer an underlying asset's risk-neutral probability (see Rubinstein 1994; Jackwerth & Rubinstein 1996; Aït-Sahalia & Lo 1998; Figlewski 2018). Since derivatives do not trade on many insurance loss events, more work is needed in this regard for the estimation of risk-neutral probabilities for loss events when markets are incomplete.

5. THE DEMAND SIDE

In this section, we review the demand side for insurance contracts by individuals in the economy. The supply side is studied in Section 6.

5.1. Consumers

A consumer's demand for insurance is a classical problem studied in economics. In general, consumers buy insurance because they are risk averse. We first illustrate this statement in a simple static model. The dynamic extensions follow.

5.1.1. Static model. We consider a consumer deciding whether or not to purchase an insurance policy at time 0 to fully protect against a possible random loss $\varepsilon > 0$ at time 1. If the insurance is purchased, the premium p is paid at time 0. We assume that the consumer satisfies the expected utility hypothesis with the objective function $E^P[U(W - \varepsilon)]$ without insurance and

[15] For example, suppose τ^i are deterministic and equal to T for all i. Here, the loss event times are trivially independent of each other and of $e^{\int_0^t r_u du}$. Then, $e^{-\int_0^{\tau^i} r_u du} 1_{\{\tau^i \leq T\}} = e^{-\int_0^T r_u du}$, which are equal and perfectly correlated across i.

$E^P[U(W-p(1+r))]$ with insurance, where $U:\mathbb{R}\to\mathbb{R}$ is the investor's utility function and W represents the consumer's random time 1 wealth without the loss event included.[16] We assume that U is twice continuously differentiable with $U' > 0$ and $U'' < 0$. These last two conditions imply that the consumer prefers more wealth to less and is risk averse. We let $r > 0$ be the default-free spot rate of interest. Finally, we assume that the random loss event ε is independent of the consumer's time 1 wealth W.

Theorem 11 (Buying insurance). The consumer will buy insurance if and only if

$$E^P\left[U(W-\varepsilon)\right] < E^P\left[U(W-p(1+r))\right]$$

if and only if

$$p < \frac{E^P[\varepsilon]}{(1+r)} + \theta,$$

where $\theta = \frac{1}{2}\frac{E^P\left[-U''(\xi_\varepsilon)(p-\varepsilon)^2\right]}{(1+r)E^P\left[U'(W-p(1+r))\right]} > 0$, $\xi_\varepsilon \in (p(1+r),\varepsilon)$ if $p(1+r) < \varepsilon$ and $\xi_\varepsilon \in (\varepsilon, p(1+r))$ if $\varepsilon < p(1+r)$.

Consumers buy insurance because they are risk averse. We see that if the premium equals its actuarial value, $p = \frac{E^P[\varepsilon]}{(1+r)}$, the consumer buys insurance. In fact, the consumer will buy insurance and pay a premium in excess of its actuarial value up to the quantity $\theta > 0$, whose magnitude depends on the consumer's wealth W, the loss ε, the premium p, the spot rate r, and the consumer's risk aversion $\frac{-U''(x)}{U'(x)}$.

In the above theorem, we assume that the insurance contract offered has full insurance coverage. Under certain conditions, it can be shown that an insurance policy with a deductible is the optimal contract design (e.g., see Arrow 1974; Borch 1983). This also holds for Pareto optimal insurance contract designs considering both the insured and the insurance company in the market structure (Raviv 1979).

It is possible that insurance coverage could change the behavior of the insured, making the loss event ε more likely to occur. For example, auto insurance could make a driver less careful when driving. This is known as moral hazard. Moral hazard does not appear in the above theorem because the decision is from the perspective of the insured, and the comparison is wealth with the possible loss versus wealth with full insurance. In the second case, ε does not appear in the consumer's preferences. For moral hazard to influence the consumer's decision, the consumer must have partial insurance, perhaps proportional insurance or a deductible.[17]

5.1.2. Dynamic extensions. At the most general and abstract level, the consumer's demand for insurance is a special case of the consumer's optimal portfolio and consumption choice problem, in which the different types of insurance contracts are some of the assets the individual can choose among. More detailed insights with respect to particular types of insurance can be obtained by specifying the contract's terms and embedding it in the portfolio and consumption problem. In this regard, life insurance, health insurance, and pension fund plans (retirement insurance) have all been studied in the literature (e.g., Richard 1975; Pliska & Ye 2007; Nielsen & Steffensen 2008; Zhou et al. 2008; Bodie, Detemple & Rindisbacher 2009; Kuhn et al. 2015).

[16] For the definitions of these terms, see Kreps (1990) or Mas-Colell, Whinston & Green (1995).
[17] For a simple example of such a model, see Varian (1978, p. 240).

5.2. Corporations and Financial Institutions

Why do firms—corporations and financial institutions—hedge? Hedging is the purchasing of insurance, often via the use of derivatives or sometimes via synthetic construction of a derivative's payoffs (e.g., a put option). To understand why, we first assume that a corporation's and financial institution's objective is to maximize the value of shareholder's equity. Next, consider a frictionless and competitive market. In such a market, there is no reason for a firm to hedge. Indeed, in such a market, the value of a firm's equity is determined by the risks of the firm's assets and liabilities. The firm can maximize the value of its equity by maximizing the value of its investments (assets) and minimizing the value of its liabilities.

Now, consider markets in which financial institutions have explicit regulatory capital constraints and all firms are subject to market frictions, in particular the dead weight losses to capital in bankruptcy. These dead weight losses are due to restricted investment choices in financial distress and third-party costs (legal fees). For financial institutions, capital constraints limit their ability to increase liabilities to fund positive-net present value investments, thereby decreasing the value of the financial institution's equity. Since a constrained (binding) optimum is always less than an unconstrained one, hedging can eliminate or reduce these regulatory capital constraints, thereby increasing the value of the financial institution's equity. For all firms (including financial institutions), hedging can eliminate or reduce the present value of bankruptcy costs, thereby increasing the value of the firm's equity. That is why firms hedge. For example, insurance companies regularly reduce regulatory capital and insolvency risk by purchasing reinsurance (see Section 6.4).

As stated earlier, hedging is the purchasing of insurance (derivatives), albeit sometimes done via synthetic construction, to reduce the risk of insolvency (bankruptcy). This is done to reduce commodity price risk, interest rate risk, foreign currency risk, and the credit risk of the firm's assets. Hedging is alternatively known as risk management. Jorion (2000) is an excellent source of information about the risk-management tools used by financial institutions from the perspective of derivatives.

6. THE SUPPLY SIDE (FUNCTIONAL APPROACH)

Using the functional approach to financial institutions (see Merton 1995), an insurance company's function in the market is to act as an intermediary providing insurance to consumers. Consequently, its primary role is to act as a broker and provide insurance without bearing the risk of the contracts on its balance sheets.[18] Unfortunately, within insurance markets (unlike equity markets), insurance companies cannot match buyers and sellers of insurance one-to-one because the demand for insurance is predominately one-sided.

This implies that to remove the insurance event loss risk from their balance sheets, insurance companies have only three alternatives that they can employ: diversification, hedging with securities/derivatives, or selling some of the loss risks in the private reinsurance market. Of these three alternatives, diversification can be used only when the event risks underlying the losses are i.i.d., but as shown in Section 4 above, this applies only to insurance products that have a yearly cycle. Otherwise, only the two remaining alternatives are available. This section presents a simple model to illustrate these assertions. We first present a static model and then discuss its dynamic extensions.

6.1. The Static Model

Assume that the insurance market is arbitrage-free, frictionless, and competitive. Let an insurance company issue products $i = 1, \ldots, n$, where the random event loss over the time period $[0, 1]$ for

[18]Insurance companies can act as dealers, too, if they so desire and can speculate on the insured events to earn a risk premium for bearing the relevant risks.

Table 1 Time 1 balance sheet

Assets	Liabilities
$ne(1+r)+\sum_{i=1}^{n}(p_i-c_i)(1+r)$	$\sum_{i=1}^{n}\varepsilon_i$

contract i is denoted $\varepsilon_i \geq 0$. The loss is paid at time 1. Assume that these losses are identically distributed with mean $\mu_\varepsilon^P > 0$ and standard deviation $\sigma_\varepsilon^P > 0$ under the probability P, where the losses are uniformly bounded with $|\varepsilon_i| \leq K$ for a constant $K > 0$.[19]

Consider an insurance company that at time 0 issues all of these insurance contracts. Each contract receives a premium of $p_i > 0$ dollars. The company invests these premiums in an mma, which earns the default-free rate $r > 0$ at time 1. It costs the company $c_i \geq 0$ dollars to issue each insurance product (an allocation of fixed costs plus marginal costs). We assume that these costs are incurred at time 0. Then, at time 1, all losses are paid to the insured by the insurance company. Therefore, the total losses paid at time 1 equal $\sum_{i=1}^{n}\varepsilon_i$. To cover its costs of operation and to guarantee that the insurance company can pay its claims, it posts an initial equity capital per contract denoted $e > 0$. This initial equity capital is also invested in the mma over [0, 1]. The insurance company's time 1 balance sheet can be found in **Table 1**.

Hence, the insurance company's time 1 equity value is

$$\mathscr{E} = ne(1+r) + \sum_{i=1}^{n}(p_i - c_i)(1+r) - \sum_{i=1}^{n}\varepsilon_i.$$

We assume that the market is arbitrage-free, but it can be incomplete (the most likely case). By Theorem 8, the insurance company selects a $Q \in \mathcal{M}$ and charges the arbitrage-free premium

$$p_i = \frac{\mu_\varepsilon^Q}{1+r} + c_i.$$

Then, the equity's time 1 value is

$$\mathscr{E} = ne(1+r) + n\mu_\varepsilon^Q - \sum_{i=1}^{n}\varepsilon_i. \tag{14}$$

The return to its equity capital is

$$\begin{aligned} R_{\mathscr{E}} &= \frac{\mathscr{E} - ne}{ne} = \frac{ner + n\mu_\varepsilon^Q - \sum_{i=1}^{n}\varepsilon_i}{ne} \\ &= r + \frac{\mu_\varepsilon^Q - \frac{\sum_{i=1}^{n}\varepsilon_i}{n}}{e}. \end{aligned}$$

Using Corollary 2, we can write this as

$$\begin{aligned} E^P(R_{\mathscr{E}}) &= r - cov^P\left(r + \frac{\mu_\varepsilon^Q - \frac{\sum_{i=1}^{n}\varepsilon_i}{n}}{e}, \frac{dQ}{dP}\right) \\ &= r + \frac{1}{n}\sum_{i=1}^{n} cov^P\left(\frac{\varepsilon_i}{e}, \frac{dQ}{dP}\right) = r + cov^P\left(\frac{\varepsilon_i}{e}, \frac{dQ}{dP}\right), \end{aligned} \tag{15}$$

[19]This section takes the form of the insurance contract as a given. Optimal insurance contract design is an important issue that is included in the discussion of equilibrium in Section 7 below.

because the losses are identically distributed. Hence, the arbitrage-free expected return to the company's equity in Equation 15 is compensation for the systematic risk of its insurance contracts. This last expression again reaffirms the conclusion that by charging arbitrage-free premiums, an insurance company's equity is properly priced relative to all other arbitrage-free prices in the market.

6.2. Capital Determination

The insured consumers and insurance regulators desire that the insurance company posts enough capital to guarantee that its insolvency risk is low and that it can pay off all insured claims with a sufficiently high probability. This can be obtained by requiring the probability that the time 1 value of the equity capital is less than or equal to zero to be small—say, less than or equal to α where $\alpha \in (0, 1)$ is a constant. In symbols, the market requires that

$$P(\mathscr{E} \leq 0) \leq \alpha.$$

By Equation 14, the probability of insolvency[20] is equivalent to

$$P\left(ne(1+r) + n\mu_\varepsilon^Q \leq \sum_{i=1}^{n} \varepsilon_i\right) \leq \alpha.$$

To discuss the issues involved, we start with the classic case, in which the losses are i.i.d.

6.2.1. Diversifiable risk (incomplete market). In the classic situation when studying an insurance company's probability of ruin and in the determination of insolvency capital, the insured random losses $\varepsilon_i \geq 0$ are i.i.d. under P (see Booth et al. 2005, chapter 14). By the law of large numbers

$$\frac{\sum_{i=1}^{n} \varepsilon_i}{n} \to \mu_\varepsilon^P,$$

with P probability 1 as $n \to \infty$. Using Theorem 9, where the premium is determined by the actuarial expected loss ($\mu_\varepsilon^Q = \mu_\varepsilon^P$), the time 1 equity value is

$$\mathscr{E} = ne(1+r) + n\mu_\varepsilon^P - \sum_{i=1}^{n} \varepsilon_i.$$

For finite but large n, by the central limit theorem

$$\frac{\frac{\sum_{i=1}^{n} \varepsilon_i}{n} - \mu_\varepsilon^P}{\left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)} \to N(0,1)$$

as $n \to \infty$, where $N(0, 1)$ is a standard cumulative normal distribution function,

$$E^P\left(\frac{\sum_{i=1}^{n} \varepsilon_i}{n}\right) = \frac{n\mu_\varepsilon^P}{n} = \mu_\varepsilon^P,$$

and

$$var^P\left(\frac{\sum_{i=1}^{n} \varepsilon_i}{n}\right) = \frac{n\left(\sigma_\varepsilon^P\right)^2}{n^2} = \frac{\left(\sigma_\varepsilon^P\right)^2}{n}.$$

[20] In the insurance literature, the probability of insolvency is known as the probability of ruin (see Melnikov 2011, p. 225).

Hence,

$$P(\mathscr{E} \le 0) = P\left(ne(1+r) + n\mu_\varepsilon^P \le \sum_{i=1}^{n} \varepsilon_i\right)$$
$$\approx N\left(\frac{e(1+r)}{\left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)} \le \frac{\frac{\sum_{i=1}^n \varepsilon_i}{n} - \mu_\varepsilon^P}{\left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)}\right).$$

Choosing

$$\frac{e(1+r)}{\left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)} = k_\alpha$$

such that

$$N\left(k_\alpha \le \frac{\frac{\sum_{i=1}^n \varepsilon_i}{n} - \mu_\varepsilon^P}{\left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)}\right) = \alpha$$

implies that the required capital necessary to satisfy the insolvency constraint is

$$e = \frac{k_\alpha \left(\frac{\sigma_\varepsilon^P}{\sqrt{n}}\right)}{1+r}. \qquad 16.$$

As $n \to \infty$, we see that due to diversification the required equity capital per insurance contract $e \to 0$. The total initial insolvency capital required is

$$ne = \frac{\sqrt{n} k_\alpha \sigma_\varepsilon^P}{1+r},$$

which grows by the square root of n. In this case, the tail risk of a large loss to the insurance company is small and controllable by its initial equity capital, which approaches zero per contract as the number of contracts approaches infinity.

6.2.2. The general case. In the general case, the identically distributed insured random losses $\varepsilon_i \ge 0$ can be correlated under P. For discussion, let us assume that as $n \to \infty$, the average insured loss converges to a random variable ξ with distribution function $F_\xi(y) = P(\xi \le y)$, i.e., with P probability one,

$$\frac{\sum_{i=1}^n \varepsilon_i}{n} \to \xi.$$

The simplest example is when the losses are perfectly correlated, in which case ξ has the same distribution as ε_i. This implies (using dominated convergence) that

$$\mu_\varepsilon^Q = E^Q\left(\frac{\sum_{i=1}^n \varepsilon_i}{n}\right) \to \mu_\xi^Q,$$

where $\mu_\xi^Q = E^Q(\xi)$. The time 1 equity value is

$$\mathscr{E} = ne(1+r) + n\mu_\varepsilon^Q - \sum_{i=1}^{n} \varepsilon_i.$$

For large n, the probability of insolvency is

$$P(\mathscr{E} \le 0) = P\left(\frac{\mathscr{E}}{n} \le 0\right) \approx P\left(e(1+r) + \mu_\varepsilon^Q \le \xi\right)$$
$$= 1 - F_\xi(e(1+r) + \mu_\xi^Q).$$

Table 2 Excess of loss reinsurance contract

Total loss L	Payoff V
$L \le K_1$	0
$K_1 < L \le K_2$	$L - K_1$
$K_2 < L$	$K_2 - K_1$

The required insolvency capital is determined by choosing e such that

$$F_\xi(e(1+r) + \mu_\xi^Q) = 1 - \alpha.$$

The solution is

$$e = \frac{F_\xi^{-1}(1-\alpha) - \mu_\xi^Q}{1+r} > 0. \qquad 17.$$

As seen, this equity capital per insurance contract does not depend on the number of contracts issued. Consequently, the insurance company's required insolvency capital $ne \to \infty$ grows proportional to n as $n \to \infty$. This is in stark contrast to the diversifiable risk case. This implies that the insurance company's tail risk is significant. When losses are correlated in this manner, to insure solvency, the insurance company needs to hedge its tail risk. It can do this by purchasing reinsurance contracts in the reinsurance market; purchasing traded derivatives (e.g., catastrophe bonds and weather derivatives); or synthetically constructing suitable reinsurance derivatives, to the extent that this is possible with traded securities in the event losses.

6.3. Dynamic Extensions

If the insurance company sells products whose maturity are multiyear, where the premiums are fixed at the time of issuance, then a dynamic model of insolvency is needed. A simple form of such a model is known as the Cramer-Lundberg model (see Melnikov 2011, chapter 8). In a dynamic setting, issues arise as to the payment of dividends and the matching of assets/liabilities in order to reduce insolvency risk (see, e.g., Waldmann 1988; Asmussen & Taksar 1997; Paulsen 1998; Hubalek & Schachermayer 2004).

6.4. Reinsurance Markets

Reinsurance is the name given to private insurance contracts purchased by insurance companies from other insurance or reinsurance companies to remove some of the risk of their losses and thereby reduce required equity capital. There are two basic types of reinsurance contracts: proportional and nonproportional reinsurance (see Booth et al. 2005, chapter 15).

With proportional reinsurance, the insurance company typically insures a fixed percent of all of its risks. Proportional insurance is often purchased by life insurance companies to increase the number of insured that they cover ($n \to \infty$) in order that the law of large numbers provides a better approximation (see Booth et al. 2005, p. 315). For nonproportional reinsurance, two common examples include stop-loss and excess of loss. These products usually provide protection for losses above some fixed level but below some cap. For example, a typical payoff to an excess of loss reinsurance contract, denoted V, is given in **Table 2**, where L is the total random loss and $0 < K_1 < K_2$ are constants. The upper limit K_2 could be infinite. This payoff is easily seen to be equivalent to the expression

$$V = \max[L - K_1, 0] - \max[L - K_2, 0],$$

which shows that an excess of loss reinsurance contract is equivalent to being long a European call option on the total loss with strike price K_1 and short a European call option on the total loss with strike price K_2. This implies typical option pricing methodologies can be used to value reinsurance contracts. Nonproportional reinsurance contracts are used to remove concentrated risk in an insurer's liabilities.

In a complete market, there is no reason for the existence of reinsurance companies. Here, the loss risks are traded and insurance companies can create synthetic reinsurance contracts by trading dynamically in the underlying traded securities. It is also possible that the loss risks trade in markets, which are completed by the trading of derivatives—examples include catastrophe bonds and weather derivatives. In this case, private reinsurance markets do not need to exist. However, most loss risks are not traded nor is there a sufficient number of relevant derivatives being traded. Under these circumstances, insurance companies depend on reinsurance companies to provide the private reinsurance.[21] For an economic critique of reinsurance markets, see Froot (2001).

6.5. Government Insurance

There are some event risks faced by consumers in a society whose losses are highly correlated and/or whose magnitudes are too large for private insurance companies and/or financial markets to cover. And, if these risks are deemed socially undesirable, there is a role for governments to provide such insurance. This is the case, for example, in the United States for bank deposits, pension funds, and health insurance. The U.S. Federal Deposit Insurance Corporation (FDIC) provides such deposit insurance to banks for a premium, which depends on the risk of the bank (see **https://www.fdic.gov**). Readers are referred to the work by Merton (1977) and Duffie et al. (2003) for models that compute arbitrage-free FDIC deposit insurance premiums and the work by Calomiris & Jaremski (2016) for a discussion of the theory and empirical evidence related to government-issued deposit insurance. A U.S. government agency, the Pension Benefit Guaranty Corporation (PBGC), provides insurance for pension funds (see **https://www.pbgc.gov**) (Bodie & Merton 1993), and the U.S. government provides health care insurance for retirees as well (see **https://www.medicare.gov**). One can think of pension plans (e.g., social security) as retirement income insurance.[22]

7. EQUILIBRIUM AND CONTRACT DESIGN

Arbitrage-free valuation for the determination of the insurance premium, as discussed above, is consistent with equilibrium as long as the existence of an equilibrium implies no arbitrage. Important issues arise in the determination of an equilibrium when the insurance company's supply is not taken as exogenous. In this case, asymmetric information—adverse selection—and how actions of the insured affect the realized losses—moral hazard—become important considerations in the optimal insurance contract designs offered by insurance companies in equilibrium. We discuss all of these issues in this section.

7.1. Exogenous Insurance Supply

This section describes the equilibrium determination of insurance premiums in a continuous-time asset pricing model. The following discussion is based on the work by Jarrow (2019). For this class of models, the supply of the available assets to buy and sell in the market is fixed and exogenous.

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[21]It is interesting to note that reinsurance companies also provide insurance to other reinsurance companies. This compounding of reinsurance coverage is called the retrocession market (see Booth et al. 2005, p. 378; Melnikov 2011, p. 203).

[22]For further discussions of these issues, see Bodie (1989) and Merton (1983).

Hence, insurance companies' supply of insurance to the market is exogenous. In addition, the payoffs to all the assets (dividends and liquidating values), including those for insurance contracts, are also fixed and exogenous to the model. Hence, an investor's actions cannot affect an asset's payoffs or an insurance policy's losses; i.e., there is no moral hazard in these models. This implies, of course, that an insurance company's optimal investment decisions are fixed and not included within this class of models.

Asset (and insurance) markets are assumed to be frictionless and competitive. By frictionless, we mean there are no transaction costs and no trading constraints (e.g., short sale constraints or margin requirements). Competitive means that investors act as price takers with respect to asset prices, including the insurance contract premiums. Asset markets need not be complete.

Investors have preferences represented by the expected utility hypothesis over consumption and terminal wealth. Utility functions are state-dependent, strictly increasing and strictly concave in consumption and terminal wealth for all states. Strictly increasing means that investors prefer more wealth to less, and strictly concave implies that investors are risk averse for independent gambles (Jarrow & Li 2021). Investors have differential beliefs and differential information. Finally, investors are endowed with an exogenous stochastic income stream.

A competitive equilibrium is defined to be an asset price (and insurance premium) process such that supply equals aggregate demand and where investor demands are optimal for the given price processes. If an equilibrium exists,[23] then the markets are arbitrage-free.[24] This follows because an investor cannot have an optimal consumption and terminal wealth process if they can become infinitely wealthy via the exploitation of an arbitrage opportunity.

The previous statement implies that the arbitrage-free insurance premiums presented in Section 4 are consistent with such a competitive equilibrium. For the arbitrage-free valuation formulas presented, equilibrium provides a characterization of the risk-neutral probabilities Q in terms of the primitives of the economy (beliefs, information, preferences, and endowments). This is the key additional insight that equilibrium provides with respect to the determination of arbitrage-free insurance premiums.

7.2. Endogenous Insurance Supply

Economists have also studied equilibrium models in which the insurance company's supply decision and the contract design are both endogenous. The classical paper on this topic is by Rothschild & Stiglitz (1976), who study markets in which insurance companies have imperfect information about the risk of the insured; hence, this is an adverse selection problem where the insured can hide their risks. There is no moral hazard in this model. In subsequent papers, adverse selection and moral hazard are studied in conjunction (Stiglitz 1983; Stiglitz & Yun 2013). In essence, these papers reveal that the optimal equilibrium contract design is constructed to both reduce moral hazard and facilitate the revelation of the insured's private information. To the extent that the insurance market equilibrium excludes arbitrage across insurance companies,[25] then the arbitrage-free insurance premium results presented in Section 4 are consistent with

[23] For a set of sufficient conditions, see Jarrow & Larsson (2018).

[24] In continuous-time asset pricing models, no arbitrage takes on the more technical definitions of "no free-lunch with vanishing risk" and "no dominance" (see Jarrow 2019, chapter 2).

[25] This is the case, for example, for many of the different market structures discussed by Rothschild & Stiglitz (1976).

these equilibrium models. We should point out, however, that with asymmetric information and moral hazard, the insurance markets will not be complete in the sense defined in Section 2. Nonetheless, the arbitrage-free insurance premiums given earlier still apply.

8. PROOFS

Proof of Corollary 1

Proof. Consider the following:

$$x = \frac{E^Q(X)}{1+r} \text{ implies } (1+r)x = E^Q(X),$$

$$rx = E^Q(X - x),$$

$$r = E^Q\left(\frac{X-x}{x}\right) = E^Q(R) = E^P\left(R \cdot \frac{dQ}{dP}\right).$$

$$\text{But, } cov^P\left(R, \frac{dQ}{dP}\right) = E^P\left(R \cdot \frac{dQ}{dP}\right) - E^P(R)E^P\left(\frac{dQ}{dP}\right).$$

Noting that $E^P(\frac{dQ}{dP}) = 1$ and substitution yields

$$r = cov^P\left(R, \frac{dQ}{dP}\right) + E^P(R).$$

Algebra completes the proof. □

Proof of Theorem 5

Proof. Consider the equally weighted portfolio

$$X^n = \frac{\sum_{i=1}^n v_i(1)}{n}, \tag{18}$$

with initial investment $x^n = \frac{\sum_{i=1}^n v_i(0)}{n}$.

Fix any $Q \in \mathcal{M}$. In an incomplete market, there are an infinite number of such Q. By the above Remark 1, however, all of these probabilities give the same arbitrage-free value for X^n, which equals x^n. Thus,

$$x^n = \frac{E^Q(X^n)}{1+r} = \frac{\mu_v^Q}{1+r}, \tag{19}$$

and the second equality holds because the assets are identically distributed.

By the law of large numbers, with P probability one, the value of the portfolio converges to its mean value, i.e.,

$$X^n = \frac{\sum_{i=1}^n v_i(1)}{n} \to \mu_v^P.$$

Using dominated convergence, expression in Equation 19 yields

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} \frac{E^Q(X^n)}{1+r} = \frac{E^Q\left(\lim_{n\to\infty} X^n\right)}{1+r} = \frac{\mu_v^P}{1+r}.$$

Equation 19 and this last equality complete the proof of Equations 5 and 6. From Remark 1, we have

$$E^P\left(\frac{X^n - x^n}{x^n}\right) = r - cov^P\left(X^n, \frac{dQ}{dP}\right) = r - cov^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}, \frac{dQ}{dP}\right) \qquad 20.$$

$$\lim_{n\to\infty} E^P\left(\frac{X^n - x^n}{x^n}\right) = r - \lim_{n\to\infty} cov^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}, \frac{dQ}{dP}\right).$$

But

$$\begin{aligned} cov^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}, \frac{dQ}{dP}\right) &= E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\frac{dQ}{dP}\right) - E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right)E^p\left(\frac{dQ}{dP}\right) \\ &= E^Q\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right) - E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right), \end{aligned}$$

since $E^p\left(\frac{dQ}{dP}\right) = 1$. Taking limits, the right side equals

$$\lim_{n\to\infty} cov^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}, \frac{dQ}{dP}\right) = \lim_{n\to\infty} E^Q\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right) - \lim_{n\to\infty} E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right).$$

Using dominated convergence, this implies

$$= E^Q\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n v_i(1)}{n}\right) - E^P\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n v_i(1)}{n}\right) = \mu_v^P - \mu_v^P = 0,$$

because convergence with probability one in P is the same as in Q since the probabilities are equivalent. Hence, the right side of Equation 20 equals r. For the left side of Equation 20,

$$E^P\left(X^n\right) = E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right) = E^P(v_i(1))$$

and

$$x^n = \frac{\sum_{i=1}^n v_i(0)}{n} = v_i(0),$$

since the assets are identically distributed. Given this is a constant sequence, we have

$$\lim_{n\to\infty} E^P\left(\frac{X^n - x^n}{x^n}\right) = E^P\left(\frac{v_i(1) - v_i(0)}{v_i(0)}\right).$$

Combining completes the proof. □

Proof of Theorem 6

Proof. Consider the equally weighted portfolio

$$X^n = \frac{\sum_{i=1}^n v_i(1)}{n} = Z + \frac{\sum_{i=1}^n \eta_i}{n},$$

with initial value

$$x^n = \frac{\sum_{i=1}^n v_i(0)}{n} = v_i(0).$$

By Remark 1,

$$x^n = \frac{E^Q(X^n)}{1+r} = \frac{E^Q(Z)}{1+r} + \frac{E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r} = \frac{z}{1+r} + \frac{\mu_\eta^Q}{1+r},$$

since the η_i are identically distributed.

Since x^n is a constant sequence,

$$\lim_{n\to\infty} x^n = z + \frac{\mu_\eta^Q}{1+r}.$$

By dominated convergence, we also have

$$\lim_{n\to\infty} x^n = \frac{E^Q(\lim_{n\to\infty} X^n)}{1+r} = \frac{E^Q(Z)}{1+r} + \frac{E^Q\left(\lim_{n\to\infty} \frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r}.$$

Using the law of large numbers, with P probability one, $\lim_{n\to\infty} \frac{\sum_{i=1}^n \eta_i}{n} = \mu_\eta^P$. Thus,

$$\lim_{n\to\infty} x^n = z + \frac{\mu_\eta^P}{1+r}$$

and

$$\mu_\eta^Q = \mu_\eta^P.$$

Since $\lim_{n\to\infty} x^n = v_i(0)$ because this is a constant sequence,

$$v_i(0) = z + \frac{\mu_\eta^P}{1+r}.$$

By Remark 1 and Corollary 1,

$$E^P\left(\frac{X^n - x}{x}\right) = r - cov^P\left(X^n, \frac{dQ}{dP}\right) = r - cov\left(Z, \frac{dQ}{dP}\right) - cov^P\left(\frac{\sum_{i=1}^n \eta_i}{n}, \frac{dQ}{dP}\right).$$

Taking limits,

$$\begin{aligned}\lim_{n\to\infty} E^P\left(\frac{X^n - x}{x}\right) &= \lim_{n\to\infty}\left(r - cov\left(Z, \frac{dQ}{dP}\right) - cov^P\left(\frac{\sum_{i=1}^n \eta_i}{n}, \frac{dQ}{dP}\right)\right)\\ &= r - cov\left(Z, \frac{dQ}{dP}\right) - \lim_{n\to\infty} cov^P\left(\frac{\sum_{i=1}^n \eta_i}{n}, \frac{dQ}{dP}\right).\end{aligned}$$

Now,

$$\begin{aligned}cov^P\left(\frac{\sum_{i=1}^n \eta_i}{n}, \frac{dQ}{dP}\right) &= E^P\left(\frac{\sum_{i=1}^n \eta_i}{n}\frac{dQ}{dP}\right) - E^P\left(\frac{\sum_{i=1}^n \eta_i}{n}\right)E^P\left(\frac{dQ}{dP}\right)\\ &= E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right) - E^P\left(\frac{\sum_{i=1}^n \eta_i}{n}\right),\end{aligned}$$

since $E^p\left(\frac{dQ}{dP}\right) = 1$. Taking limits,

$$\lim_{n\to\infty} cov^P\left(\frac{\sum_{i=1}^n \eta_i}{n}, \frac{dQ}{dP}\right) = \lim_{n\to\infty} E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right) - \lim_{n\to\infty} E^p\left(\frac{\sum_{i=1}^n \eta_i}{n}\right).$$

Using dominated convergence,

$$= E^Q\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n \eta_i}{n}\right) - E^p\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n \eta_i}{n}\right) = \mu_\eta^P - \mu_\eta^P = 0,$$

because convergence with probability one in P is the same as probability one convergence in Q, since they are equivalent probabilities.

For the left side,

$$E^P\left(X^n\right) = E^P\left(\frac{\sum_{i=1}^n v_i(1)}{n}\right) = E^P(v_i(1))$$

and

$$x^n = \frac{\sum_{i=1}^n v_i(0)}{n} = v_i(0),$$

since identically distributed. Because this is a constant sequence, taking the limit, we have

$$\lim_{n\to\infty} E^P\left(\frac{X^n - x^n}{x^n}\right) = E^P\left(\frac{v_i(1) - v_i(0)}{v_i(0)}\right).$$

Combined, we have

$$E^P(R_v) = r - cov^P\left(Z, \frac{dQ}{dP}\right).$$

□

Proof of Theorem 7

Proof. The time 0 value of an insurance contract is zero. In an arbitrage-free and complete market,

$$0 = \frac{E^Q\left(I_i\right)}{1+r} = \frac{E^Q\left(\left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i + c_i\right)(1+r) - c_i(1+r) - \varepsilon_i\right)}{1+r}$$
$$= \left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i\right) - \frac{E^Q(\varepsilon_i)}{1+r}.$$

Solving for the arbitrage-free premium gives

$$\pi_i = \frac{E^Q(\varepsilon_i) - E^P(\varepsilon_i)}{1+r} \geq 0.$$

The risk-premium adjustment is nonnegative, because these represent risks that consumers desire to remove. Thus,

$$p_i = \frac{E^P(\varepsilon_i) + E^Q(\varepsilon_i) - E^P(\varepsilon_i)}{1+r} + c_i = \frac{E^Q(\varepsilon_i)}{1+r} + c_i.$$

This completes the proof. □

Proof of Theorem 8

Proof. Let δ_i be the arbitrage-free value of the insurance contract for a fixed π_i. By Theorem 4, we have

$$\delta_i \in \left[\inf_{Q\in\mathcal{M}}\left\{\left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i\right) - \frac{E^Q(\varepsilon_i)}{1+r}\right\}, \inf_{Q\in\mathcal{M}}\left\{\left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i\right) - \frac{E^Q(\varepsilon_i)}{1+r}\right\}\right],$$

where $\mathrm{E}^Q\left(I_i\right)\dfrac{}{1+r = \left(\frac{E^P(\varepsilon_i)}{1+r} + \pi_i\right) - \frac{E^Q(\varepsilon_i)}{1+r}}.$

This equals

$$\left[\frac{E^P(\varepsilon_i)}{1+r}+\pi_i-\sup_{Q\in\mathcal{M}}\left(\frac{E^Q(\varepsilon_i)}{1+r}\right),\frac{E^P(\varepsilon_i)}{1+r}+\pi_i-\inf_{Q}\left(\frac{E^Q(\varepsilon_i)}{1+r}\right)\right].$$

Setting the value of the insurance contract to zero gives the range of arbitrage-free insurance risk-premium adjustments as

$$\pi_i\in\left[\inf_{Q\in\mathcal{M}}\left\{\frac{E^Q(\varepsilon_i)-E^P(\varepsilon_i)}{1+r}\right\},\sup_{Q\in\mathcal{M}}\left\{\frac{E^Q(\varepsilon_i)-E^P(\varepsilon_i)}{1+r}\right\}\right]$$

and arbitrage-free premiums as

$$\begin{aligned}p_i&\in\left[\inf_{Q\in\mathcal{M}}\left\{\frac{E^Q(\varepsilon_i)}{1+r}+c_i\right\},\sup_{Q\in\mathcal{M}}\left\{\frac{E^Q(\varepsilon_i)}{1+r}+c_i\right\}\right]\\&=\left[\inf_{Q\in\mathcal{M}}\left\{\frac{\mu_\varepsilon^Q}{1+r}+c_i\right\},\sup_{Q\in\mathcal{M}}\left\{\frac{\mu_\varepsilon^Q}{1+r}+c_i\right\}\right].\end{aligned}$$

This completes the proof. □

Proof of Theorem 9

Proof. Fix any $Q\in\mathcal{M}$. In an incomplete market, by Theorem 4, this gives the arbitrage-free premium for the derivative with payoff $\varepsilon_i\geq 0$ as

$$p_i=\frac{\mu_\varepsilon^Q}{1+r}.$$

Now, consider the portfolio of derivatives $\frac{\sum_{i=1}^n\varepsilon_i}{n}$. The arbitrage-free value for this portfolio under Q is

$$\frac{E^Q\left(\frac{\sum_{i=1}^n\varepsilon_i}{n}\right)}{1+r}=\frac{\mu_\varepsilon^Q}{1+r}.$$

This is a constant sequence, hence

$$\frac{\lim_{n\to\infty}E^Q\left(\frac{\sum_{i=1}^n\varepsilon_i}{n}\right)}{1+r}=\frac{\mu_\varepsilon^Q}{1+r}.$$

But, using dominated convergence,

$$\frac{\lim_{n\to\infty}E^Q\left(\frac{\sum_{i=1}^n\varepsilon_i}{n}\right)}{1+r}=\frac{E^Q\left(\lim_{n\to\infty}\frac{\sum_{i=1}^n\varepsilon_i}{n}\right)}{1+r}=\frac{\mu_\varepsilon^P}{1+r}.$$

Hence,

$$\mu_\varepsilon^Q=\mu_\varepsilon^P.$$

Since $Q\in\mathcal{M}$ was arbitrary, the premium is uniquely determined by its actuarial value, i.e.,

$$p_i=\frac{\mu_\varepsilon^P}{1+r}+c_i.$$

This completes the proof. □

Proof of Theorem 10

Proof. Fix any $Q \in \mathcal{M}$. By Theorem 4, the arbitrage-free premium for this $Q \in \mathcal{M}$ is

$$p_i = \frac{E^Q(\varepsilon_i)}{1+r}.$$

Substitution yields

$$p_i = \frac{E^Q(Z)}{1+r} + \frac{\mu_\eta^Q}{1+r}.$$

But, recall that $z = \frac{E^Q(Z)}{1+r}$. Thus,

$$p_i = z + \frac{\mu_\eta^Q}{1+r}.$$

Now, consider the portfolio of derivatives $\frac{\sum_{i=1}^n \eta_i}{n}$. This portfolio can be obtained by holding $\frac{\sum_{i=1}^n \varepsilon_i}{n}$ and shorting the payoff Z. Given Q, its arbitrage-free value is

$$\frac{E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r} = \frac{\mu_\eta^Q}{1+r},$$

because the η_i are identically distributed. This is a constant sequence, hence

$$\frac{\lim\limits_{n\to\infty} E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r} = \frac{\mu_\eta^Q}{1+r}.$$

Now, using dominated convergence, and by the law of large numbers since $\lim\limits_{n\to\infty} \frac{\sum_{i=1}^n \eta_i}{n} \to \mu_\eta^P$ with P probability one, we gave

$$\frac{\lim\limits_{n\to\infty} E^Q\left(\frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r} = \frac{E^Q\left(\lim\limits_{n\to\infty} \frac{\sum_{i=1}^n \eta_i}{n}\right)}{1+r} = \frac{\mu_\eta^P}{1+r},$$

which implies

$$\mu_\eta^Q = \mu_\eta^P.$$

Hence,

$$p_i = z + \frac{\mu_\eta^P}{1+r} + c_i.$$

This completes the proof. □

Proof of Theorem 11

Proof. Using a Taylor series expansion (see Guler 2010, chapter 1), we have

$$\begin{aligned} U(W-\varepsilon) &= U(W - p(1+r) + p(1+r) - \varepsilon) \\ &= U(W-p(1+r)) + U'(W-p(1+r))(p(1+r)-\varepsilon) + \frac{1}{2}U''(\xi_\varepsilon)(p(1+r)-\varepsilon)^2 \end{aligned}$$

for $\xi_\varepsilon \in (p(1+r), \varepsilon)$ if $p(1+r) < \varepsilon$ or $\xi_\varepsilon \in (\varepsilon, p(1+r))$ if $\varepsilon < p(1+r)$. Taking expectations, using independence of $W$and ε, yields

$$\begin{aligned} E^P\left[U(W-\varepsilon)\right] = E^P\left[U(W-p(1+r))\right] + E^P\left[U'(W-p(1+r))\right]E^P\left[(p(1+r)-\varepsilon)\right] \\ + \frac{1}{2}E^P\left[U''(\xi_\varepsilon)(p(1+r)-\varepsilon)^2\right]. \end{aligned}$$

The consumer chooses to buy insurance if and only if

$$E^P\left[U(W-\varepsilon)\right] < E^P\left[U(W-p(1+r))\right],$$

$$E^P\left[U'(W-p(1+r))\right]E^P\left[(p(1+r)-\varepsilon)\right]+\frac{1}{2}E^P\left[U''(\xi_\varepsilon)(p(1+r)-\varepsilon)^2\right] < 0,$$

$$E^P\left[(p(1+r)-\varepsilon)\right] < \frac{1}{2}\frac{E^P\left[-U''(\xi_\varepsilon)(p-\varepsilon)^2\right]}{E^P\left[U'(W-p(1+r))\right]},$$

$$p < \frac{E^P[\varepsilon]}{(1+r)}+\frac{1}{2}\frac{E^P\left[-U''(\xi_\varepsilon)(p-\varepsilon)^2\right]}{(1+r)E^P\left[U'(W-p(1+r))\right]}.$$

This completes the proof. □

Proof that Diversification Fails in a Dynamic Setting

Proof. We now investigate whether the diversification argument can be extended in the dynamic setting. For the start, let us suppose that interest rates are deterministic. Clearly, this is an invalid assumption. But, it is useful to clarify why, in general, the diversification argument fails. When interest rates are deterministic, the random variables $e^{-\int_0^{\tau^i} r_u du}1_{\{\tau^i\leq T\}}$ are independent, because the only randomness is due to the loss event time.

Consider the average of a single term in the denominator of Equation 13:

$$\frac{\sum_{i=1}^n E^Q\left[1_{\{\tau^i>t_k\}}e^{-\int_0^{t_k} r_u du}\right]}{n} = B(0,t_k)\frac{\sum_{i=1}^n Q\left(\tau^i>t_k\right)}{n} = B(0,t_k)Q\left(\tau^1>t_k\right),$$

because the event times are identically distributed and $B(0,t)=E^Q\left[e^{-\int_0^t r_u du}\right]=e^{-\int_0^t r_u du}$. Taking limits (a constant sequence) yields

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n E^Q\left[1_{\{\tau^i>t_k\}}e^{-\int_0^{t_k} r_u du}\right]}{n} = B(0,t_k)Q\left(\tau^1>t_k\right).$$

By the law of large numbers,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n 1_{\{\tau^i>t_k\}}}{n} = P(\tau^1>t_k).$$

Hence,

$$\lim_{n\to\infty} e^{-\int_0^{t_k} r_u du}\frac{\sum_{i=1}^n 1_{\{\tau^i>t_k\}}}{n} = e^{-\int_0^{t_k} r_u du}P(\tau^1>t_k).$$

By dominated convergence,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{i=1}^n E^{\mathbb{Q}}\left[e^{-\int_0^{t_k} r_u du}1_{\{\tau^i>t_k\}}\right]}{n} &= E^Q\left[e^{-\int_0^{t_k} r_u du}\lim_{n\to\infty}\frac{\sum_{i=1}^n 1_{\{\tau^i>t_k\}}}{n}\right]\\ &= E^Q\left[e^{-\int_0^{t_k} r_u du}\right]P(\tau^1>t_k) = B(0,t_k)P(\tau^1>t_k).\end{aligned}$$

This implies that $Q(\tau^i>t_k)=P(\tau^i>t_k)$ for all i.

Next, consider the term in the numerator of Equation 13:

$$E^Q\left[1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}\right].$$

By the law of large numbers, this converges to its mean under P, i.e.,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n 1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}}{n}=E^P\left[1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}\right].$$

Consider the average

$$\frac{\sum_{i=1}^n E^Q\left[1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}\right]}{n}=E^Q\left[1_{\{\tau^1\le T\}}e^{-\int_0^{\tau^1}r_u du}\right],$$

which follows because the event times are identically distributed. As a constant sequence,

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n E^Q\left[1_{\{\tau^i\le T\}}e^{-\int_0^{t_k}r_u du}\right]}{n}=E^Q\left[1_{\{\tau^1\le T\}}e^{-\int_0^{\tau^1}r_u du}\right].$$

By dominated convergence,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{i=1}^n E^Q\left[1_{\{\tau^i\le T\}}e^{-\int_0^{t_k}r_u du}\right]}{n}&=E^Q\left[\lim_{n\to\infty}\frac{\sum_{i=1}^n e^{-\int_0^{t_k}r_u du}1_{\{\tau^i\le T\}}}{n}\right]\\&=E^Q\left[E^P\left[e^{-\int_0^{\tau^i}r_u du}1_{\{\tau^i\le T\}}\right]\right]=E^P\left[1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}\right].\end{aligned}$$

That is,

$$E^Q\left[1_{\{\tau^1\le T\}}e^{-\int_0^{\tau^1}r_u du}\right]=E^P\left[1_{\{\tau^1\le T\}}e^{-\int_0^{\tau^1}r_u du}\right].$$

Substitution into Equation 13 yields

$$c=\frac{LE^P\left[1_{\{\tau^i\le T\}}e^{-\int_0^{\tau^i}r_u du}\right]}{\sum_{k=1}^m B(0,t_k)P(\tau^i>t_k)}.$$

Under these assumptions, the arbitrage-free insurance premium is determined by its actuarial fair value.

Unfortunately, when interest rates are stochastic, the random variables $e^{-\int_0^{\tau^i}r_u du}1_{\{\tau^i\le T\}}$ for $i=1,\ldots,m$ are no longer independent. So the law of large numbers cannot be invoked to replace Q with P, and the diversification argument in an incomplete market fails. □

9. CONCLUSION

This article reviews the economics of insurance literature from the perspective of financial economics, emphasizing the importance of derivatives' pricing and hedging. In the process, some new insights are obtained with respect to insurance premium determination and managing an insurance company's insolvency risk.

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