ANNUAL REVIEWS

Annual Review of Financial Economics

Inflation-Adjusted Bonds, Swaps, and Derivatives

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Annu. Rev. Financ. Econ. 2023. 15:449–71

First published as a Review in Advance on June 20, 2023

The *Annual Review of Financial Economics* is online at financial.annualreviews.org

https://doi.org/10.1146/annurev-financial-110921-110855

JEL codes: G12, G13

Keywords

inflation-indexed bonds, inflation swaps, inflation derivatives, HJM model, term structure of interest rates, break-even inflation rate

Abstract

The purpose of this article is to review the literature on inflation-adjusted bonds, swaps, and derivatives. The methodology for valuation and risk management of these securities is an application of the foreign currency extension of a standard HJM term structure model. The two "currencies" in the extended model are real and nominal prices. Currently, for their use in monetary policy, the empirical literature primarily uses these models to estimate both the expected inflation rate and the inflation risk premium. A literature investigating the efficiency of the inflation derivative markets and a comparison of the relevant valuation models is almost nonexistent and a fruitful area for future research.

1. INTRODUCTION

The purpose of this article is to review the literature on inflation-adjusted (-linked) bonds, inflation swaps, and inflation derivatives. To understand inflation-adjusted bonds, it is necessary to first characterize the arbitrage-free relationship between the term structure of nominal bonds and the term structure of inflation-adjusted bonds. This characterization is an application of the foreign currency (FX) extension of a standard HJM term structure model (Heath, Jarrow & Morton 1992; see also Amin & Jarrow 1991). The two "currencies" in such an extended HJM model are real and nominal prices. Given this FX analogy, the standard valuation and risk management methodologies employed in the derivative pricing literature apply. These standard methodologies enable a complete understanding of inflation-adjusted bonds, inflation swaps, inflation futures, and more complex inflation derivatives.

Given this theory, the empirical literature primarily uses these models to estimate both the expected inflation rate over various future horizons and the inflation risk premium. Expected inflation rates are useful for central banks in their formulation of monetary policy. Estimates of the inflation risk premium are beneficial to governments in deciding whether to finance government deficits by issuing either nominal or inflation-adjusted bonds. For example, if the inflation risk premium is positive, then issuing inflation-adjusted bonds that reflect no inflation risk premium provides a cheaper borrowing instrument than issuing nominal bonds. In addition, if there is a positive inflation risk premium, then by helping to complete the market, investors' welfares are improved by the issuance of inflation-adjusted bonds. Surprisingly, a related empirical literature either investigating the efficiency of the inflation derivative markets or providing a comparison of the valuation models is almost nonexistent and a fruitful area for future research.

An outline of this article is as follows. Sections 2 and 3 discuss the inflation-adjusted bonds and various traded inflation derivatives, respectively. Section 4 presents the basic arbitrage-free valuation model, and Section 5 applies it to derive the Fisher equation and the break-even inflation rate (BEIR). Sections 6 and 7 value coupon bonds and inflation swaps, respectively. Section 8 discusses arbitrage between bonds and swaps. Section 9 studies options, Section 10 reviews the empirical literature on inflation-adjusted bonds, while Section 11 concludes with some directions for future research.

2. INFLATION-ADJUSTED BONDS

Conventional bonds are nominal bonds, paying coupons based on a fixed rate, whose real value declines with inflation (Bekaert & Wang 2010). In contrast, inflation-adjusted bonds are designed to help protect investors from the risks of inflation and are typically issued by governments.[1] By construction, they guarantee a (nearly) fixed real return. Even in low and stable inflationary periods, the low volatility of real returns and their low correlation with other asset returns make them attractive for portfolio diversification.

In the construction of these inflation-adjusted bonds, a basket of predefined expenditures is used to measure inflation. Although the weights of the items in the basket differ across countries, the three largest components are housing, transportation, and food. To offer exact inflation protection, coupon and principal repayments to the inflation-adjusted bonds would need to be adjusted with respect to realized inflation. However, this is typically not the case, because inflation indices are computed with a lag. For example, in the United States, the inflation-adjusted

[1] Historically, real estate was considered a good inflation hedge, but recent evidence is mixed in this regard (Hoesli, Lizieri & MacGregor 2008).

bonds are called Treasury Inflation-Protected Securities (TIPS or TIIS), and the inflation index is the nonseasonally adjusted US city average of all items Consumer Price Index for All Urban Consumers (CPI-U) published by the Bureau of Labor Statistics of the US Department of Labor. There is a 2- to 3-month lag in compiling the CPI. Therefore, to approximate the actual realized inflation, daily CPI levels are linearly extrapolated to create realized daily adjustments.

Currently, 20 countries regularly issue inflation-adjusted bonds, and the total notional volume in February 2022 was approximately 4.4 trillion dollars (Ranasinghe, Chatterjee & Barbuscia 2022). The Commonwealth of Massachusetts in the United States issued the earliest recorded inflation-indexed bonds in 1780 during the Revolutionary War (see Shiller 2005). Modern governments began issuing inflation-linked bonds (ILBs) much later, spurred on by high inflation in conjunction with regulatory and tax changes.

In the 1980s, a period of high inflation, the United Kingdom was a major issuer of these bonds. With the UK pension reform of 1984, pension companies became the leading investors in these securities to mitigate unexpectedly large inflation. In the late 1990s, more countries stepped into the market. The United States issued 5- and 10-year TIPS in 1997 and 30-year TIPS in 1998. With the reform of tax-exempt saving accounts in France, the French inflation-linked Obligation Assimilable du Tresor (OATi) was launched in 1998. Later, Italy in 2003, Germany in 2005, and others in Europe issued such bonds linked to the European Monetary Union Harmonised Index of Consumer Prices (EMU HICP). Following the UK Retail Price Index (RPI) and the US Consumer Price Index (CPI), EMU HICP is the third major index for inflation securities. Although Uruguay, Chile, Brazil, Israel, and the United Kingdom's shares of the inflation-bond market in their overall debt issues exceed 20%, TIPS account for less than 10% of the US government debt market. Yet, TIPS represent the largest market notional volume in the global market, reaching a total market value of 1.5 trillion dollars in December 2021 (BlackRock 2023), and they are growing with increased inflationary expectations.

The US federal government offers another type of inflation-adjusted bond called a US Series I Saving Bond. To determine the interest paid on TIPS, the interest calculations are based upon an inflation-adjusted principal using the CPI-U. The principal of a TIPS bond increases with inflation and decreases with deflation, although it can never fall below the bond's original principal. In this regard, the bond embeds an inflation put option. However, the embedded put option has had little value (if any), since the CPI-U has continuously increased [similarly, the embedded call option has little value with CPI-U decreases (see Jarrow & Yildirim 2003)]. In contrast, the earnings rate of a Series I Bond is determined as the sum of a fixed rate of return, set at the time of purchase, and a variable semiannual inflation rate. TIPS are issued with 5-, 10-, and 30-year maturities, while the Series I Bonds only have a 30-year maturity.

TIPS can be purchased online through Treasury Direct, a website run by the Bureau of the Fiscal Service under the US Department of the Treasury, or over-the-counter through a bank or broker. Both TIPS and US Series I Bonds are subject to federal tax but exempt from state and local income taxes. However, they differ in the timing of federal tax payment; the semiannual interest payments and inflation adjustments that increase the principal of TIPS are subject to federal tax in the year that they occur, while the tax reporting of interest payments can be deferred until redemption or final maturity for the Series I Bonds.[2]

The connection between the nominal yield (i.e., conventional bond yield) and the real yield (i.e., the yield in excess of expected inflation) is through the Fisher equation, as discussed in Section 5. The nominal yield consists of the real yield, expected inflation, and an inflation risk

[2]For more information, see **ftp://ftp.publicdebt.treas.gov/gsrintax.pdf**.

premium. The BEIR, a closely watched indicator of inflation, is defined to be the difference between the nominal and real yield. Interestingly, inflation-adjusted bonds can have negative yields, specifically when the bond's yields are below the expected inflation rate. For example, TIPS traded with negative yields in late 2011. TIPS yields turned positive between June 2013 and February 2020 but dropped to negative again during the COVID-19 pandemic.

3. INFLATION DERIVATIVES

Financial institutions, such as pension funds and insurance companies, have inflation-adjusted long-term obligations to their clients. But, their cash inflows are often fixed. Since future inflation rates are unknown, this generates inflation risk on their balance sheets. Purchasing inflation-adjusted bonds provides a partial hedge for this risk, but maturity mismatches between the liability payment dates and the bonds' maturities may still exist. Inflation derivatives can be used to eliminate such problems. Historically, as the inflation-adjusted bond markets became more liquid and traders more sophisticated, financial intermediaries started offering more tailor-made inflation swaps to help clients hedge their unique inflation-linked exposures.

Although the UK market introduced the first inflation derivatives after the gilt repo market began in 1996, the larger market growth did not start until after 2003. In the global market, the traded notional volume of inflation swaps in 2021 hit a record \$1.67 trillion, increasing by 37% from \$1.21 trillion in 2020. The trade count of inflation swaps also grew rapidly by 31%, from 34,198 in 2020 to 44,896 in 2021. The monthly cleared notional volume in inflation swaps is now over \$500 billion, reaching that for the first time in March 2021.[3] As inflationary expectations have increased, volumes have grown larger. On a broader level, the Interest Rate Derivatives (IRD) traded notional and trade count grew by 0.5% and 19.5%, respectively, in 2021 versus the year before. Based on the January 2022 report by the International Swaps and Derivatives Association (ISDA), the IRD traded notional value increased to \$231.0 trillion in 2021 from \$229.7 trillion in 2020, while the trade count rose to 1.9 million from 1.6 million over the same period (Int. Swaps Deriv. Assoc. 2022).

Many kinds of inflation derivatives are available over-the-counter, such as inflation swaps, inflation options, inflation-adjusted bond options, inflation-linked equity, inflation-linked equity options, inflation-linked credit default swaps, and inflation-linked collateralized debt obligations (CDOs). We now describe the common inflation derivatives and their uses for hedging purposes.

3.1. Inflation Swaps

There are two main inflation-adjusted swaps traded in the market: the zero-coupon and the year-on-year swaps (Benaben & Goldenberg 2008, Brigo & Mercurio 2006).

A zero-coupon inflation swap is simply a forward rate contract on the future inflation rate. Protection buyers (e.g., pension funds) agree to pay a compounded fixed-rate payment on the notional amount at the maturity. In contrast, the protection seller (e.g., an investment bank) agrees to pay the inflation-adjusted notional payment at maturity, determined by the accrued inflation. The protection buyer transfers the risk to the seller via the swap contract when hedging an inflation-adjusted liability. On the other side, the protection seller who assumes this risk can in turn hedge its inflation risk, if desired, by using inflation-adjusted bonds (see Section 8 for details on how to do this). As shown in Section 7, these contracts are known as break-even inflation swaps because the swap fixed rate equals the BEIR on the trade under no-arbitrage.

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[3] For inflation swap activities between 2005 and 2012 in the United States, see Fleming & Sporn (2013).

A year-on-year inflation swap is an agreement to exchange fixed-rate payments against inflation-adjusted coupon payments every year, instead of the one-time exchange of cash flows at the maturity, as in the case of a zero-coupon inflation swap. The fixed rate is the quoted rate. The hedging uses are the same as for the zero-coupon inflation swap.

3.2. Inflation Options

The most popular inflation options are caps/floors, options on caps/floors, and swaptions. Inflation caps/floors provide protection against the CPI being higher/lower than an agreed upon strike. They consist of a series of caplets/floorlets. For example, a 5-year inflation cap/floor with an annual adjustment has a caplet/floorlet that matures after the first year, a caplet/floorlet that matures after 2 years, and so on.

Inflation caplets/floorlets are call/put options on the inflation rate implied by the inflation index, such as CPI. They provide buyers with protection against upside/downside risks. Since they are not often traded, their prices are implied from the observed caps/floors prices.

Inflation swaptions are another popular product. As noted earlier, inflation swap contracts are binding agreements. However, the protection buyer hedging inflation risk may like to enter, change the size, or cancel the inflation swap contract with a predefined cost. For example, new tax regulations and mortality rates may change a pension fund's liabilities. Hedging such events, they can purchase inflation swaptions, a contract that gives the right to enter into an inflation swap at a prespecified time with a fixed strike. The buyer will exercise the option to enter into the contract if the inflation swap has a positive value.

3.3. Hybrid Products

Investors hedge inflation risk exposures of their liabilities with inflation-adjusted bonds, swaps, or options, as just described. However, investors may also have other assets, such as equity and credit-linked assets, in their portfolios whose returns are also exposed to the risk of inflation. Some hybrid products allow inflation-adjusted cash flows in payoff schemes generating real equity or real credit-linked returns. As the name suggests, these products combine a contract with embedded inflation derivatives.

Inflation-adjusted equity is a structured product whose return is linked to the inflation-adjusted performance of its underlying equity. They are debt instruments (i.e., notes) split into their notional and interest components. An inflation-adjusted zero-coupon bond protects the notional amount. At the maturity of the note, the final payout to the investor will be the inflation-adjusted principal plus the inflation-adjusted payoff on the equity options. The Italian post office issued the first such product in early 2000.

An inflation spread option derives its value from the difference between two inflation indices, for example, the EMU HICP and the UK RPI. The protection buyers hedge against the risk that the index is used for hedging inflation exposures diverging from the benchmark index. If the benchmark is an untraded index, the protection seller will ask for a higher premium when selling the option.

3.4. Inflation Futures

Inflation futures were first introduced in the United States in 1997 when the Chicago Board of Trade listed 5-year and 10-year TIPS futures contracts, later adding 30-year TIPS futures in 1998. However, they were delisted in 2001 due to the declining risk of inflation. Afterward, the Chicago Mercantile Exchange (CME) launched US CPI futures in 2004. Investors can use CPI futures to express their views on future inflation. One can use these CPI futures to set up a hedge against a TIPS cash flow, allowing the creation of tradeable forward real yields.

4. THE BASIC VALUATION MODEL

To discuss valuation and hedging, we construct a general arbitrage-free market using the FX analogy introduced by Jarrow & Yildirim (2003). We consider a continuous trading and continuous time model on the finite interval $t \in [0, \mathscr{T}]$, where $\mathscr{T}$ denotes the model's horizon. We are given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\mathscr{T}]}$ satisfies the usual hypotheses and $\mathcal{F} = \mathcal{F}_{\mathscr{T}}$.[4] Here, $\mathbb{P}$ is the statistical probability measure. By the statistical probability measure $\mathbb{P}$, we mean that from which historical time series data are generated (drawn by nature). Hence, standard statistical methods can be used to estimate this probability measure $\mathbb{P}$ from historical time series data.

Traded in a frictionless and competitive market are nominal and inflation-protected zero-coupon bonds of all maturities and a nominal money market account (details provided in Section 4.1). By frictionless, we mean that there are no transaction costs, there are no differential taxes, shares are infinitely divisible, and there are no trading constraints (e.g., short sales restrictions, borrowing limits, or margin requirements). By competitive, we mean that traders act as price takers, i.e., they can trade any quantity of shares desired without affecting the market price. Alternatively stated, there is no liquidity risk. Liquidity risk is when there is a quantity impact from trading on the price (for additional discussion, see Cetin, Jarrow & Protter 2004).

4.1. The Bonds

As noted, nominal default-free zero-coupon bonds of all maturities are traded, paying a sure nominal dollar 1_n at time T, with time t prices denoted $P_n(t, T) > 0$ for $0 \le t \le T \le \mathscr{T}$. The indexing implies the existence of zero-coupon bonds of all maturities up to time $\mathscr{T}$ for all dates t, where $t \le T$ and T corresponds to the bond's maturity. The zero-coupon bond prices are assumed to be strictly positive to avoid trivial mispricings in the market. The subscript n denotes nominal. A nominal money market account (mma) also trades with time t value given by

$$B(t) = e^{\int_0^t r(s)ds},$$

where $r(s)$ is the default-free spot rate at time s. We assume that the stochastic processes $r(t)$, $P_n(t, T)$ are semimartingales adapted to $\mathbb{F}$. As such, these processes allow discontinuous sample path processes.

For conceptualizing the model, we first define real default-free zero-coupon bonds of all maturities paying a sure real dollar 1_r at time T, with time t price in real dollars denoted $P_r(t, T)$ for $0 \le t \le T \le \mathscr{T}$. The subscript r denotes real, i.e., units of the consumption good. We assume that the stochastic processes $P_r(t, T)$ are semimartingales adapted to $\mathbb{F}$.

An inflation index exists at time t, denoted $I(t)$ with $I(0) = 1$. This is the exchange rate of real dollars to nominal dollars, i.e., it is the number of nominal dollars each real dollar is worth at time t. We assume that the stochastic process $I(t)$ is a semimartingale adapted to $\mathbb{F}$.

Finally, inflation-protected default-free zero-coupon bonds of all maturities valued in dollars are traded, paying a sure $I(T) \cdot 1_r$ dollars at time T, with time t price in nominal dollars denoted $P_I(t, T)$ for $0 \le t \le T \le \mathscr{T}$, where

$$P_I(t, T) = I(t)P_r(t, T). \tag{1}$$

[4]For the usual conditions, see Protter (2005).

Hence, in terms of modeling, considering an economy with trading in the nominal and real bonds versus one with trading in the nominal and inflation-protected bonds is equivalent.

For later use, we implicitly define the yields on nominal and real bonds via the expressions

$$P_n(t,T) = 1_n e^{-y_n(t,T)(T-t)}$$
$$P_r(t,T) = 1_r e^{-y_r(t,T)(T-t)},$$

where $y_n(t,T), y_r(t,T)$ are the yields on the nominal and real zero-coupon bonds, respectively. Note that because the nominal payoff to an inflation-protected zero-coupon bond is random in nominal dollars, its yield is not well-defined. Instead, the yield on an inflation-protected zero-coupon bond is implicitly defined via expression (Equation 1) as

$$P_I(t,T) = I(t) 1_r e^{-y_r(t,T)(T-t)},$$

hence, its yield is equivalent to the real yield $y_r(t,T)$.

4.2. Arbitrage-Free Conditions

To understand the arbitrage-free valuation of various inflation-protected securities, including coupon bonds, we introduce the structure necessary to invoke the first and second fundamental theorems of asset pricing. We only sketch the necessary structure, leaving the formal details to Jarrow (2021, chapter 2).

Investors are allowed to hold, in a dynamic portfolio, a finite number of the traded securities. The holdings in the zero-coupon bonds must be predictable processes, and the position in the mma is an optional process. These measurability conditions are imposed to ensure that various stochastic integrals underlying the theory are well defined. This dynamic portfolio is called a trading strategy. The trading strategy's value must be uniformly bounded below, implying a borrowing constraint. Such a trading strategy is called admissible. Admissibility excludes doubling strategies [for further discussion, see Jarrow (2021, chapter 2)].

An arbitrage opportunity is any admissible self-financing trading strategy that has zero initial investment and, for some future time T, has nonnegative payoffs for sure and strictly positive payoffs with positive probability. With continuous trading, near-arbitrage opportunities must also be excluded, called free lunch with vanishing risk (FLVR) trading strategies.

The first fundamental theorem of asset pricing states that the market satisfies no free lunch with vanishing risk (NFLVR) if and only if there exists an equivalent local martingale measure $\mathbb{Q}$ for the traded zero-coupon bonds, i.e.,

$$\frac{P_n(t,T)}{B(t)}, \frac{P_I(t,T)}{B(t)}$$

for all $0 \le t \le T \le \mathscr{T}$ are $\mathbb{Q}$ - local martingales.

Equivalence means that the probabilities $\mathbb{Q}$ and $\mathbb{P}$ agree on zero-probability events in the $\sigma-$algebra $\mathcal{F}$. Formally, this implies that $\mathbb{Q}$ and $\mathbb{P}$ are related by a density process $\frac{d\mathbb{Q}}{d\mathbb{P}}$ via the relation $\mathbb{Q} = \frac{d\mathbb{Q}}{d\mathbb{P}}\mathbb{P}$. This relation gives the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ the interpretation of a risk premium for both real and inflation risk. $\mathbb{Q}$ is typically called a risk-neutral probability. Note that conditional expectation of the density $E_t^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) =: \frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ is a $\mathbb{P}$ martingale where $E_0^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \frac{d\mathbb{Q}}{d\mathbb{P}}(0) = 1$ because $\mathbb{Q}$ is a probability. The ratio $\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}{B(t)}$ is called the state price density because $P_n(t,T)\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}{B(t)}$ and $P_I(t,T)\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}{B(t)}$ are $\mathbb{P}$ martingales, implying that it is an adjustment for both risk and discounting.

A local martingale is a martingale on a sequence of stopping times approaching $\mathscr{T}$, a modest generalization of a martingale that allows for the existence of price bubbles in an NFLVR market [for more discussion, see Jarrow (2021, chapter 3)].

If we assume bond prices are bounded above, or forward rates are bounded below by some large negative number, then this implies these processes are $\mathbb{Q}$ - martingales. Or alternatively, we can assume the market satisfies no dominance (see Jarrow & Larsson 2012). Assuming that the market satisfies either of these two conditions, we have that the following martingale relations hold:

$$P_n(t,T) = E_t^{\mathbb{Q}}\left(\frac{P_n(t+\Delta,T)}{B(t+\Delta)}\right)B(t) = E_t^{\mathbb{Q}}\left(\frac{1_n}{B(T)}\right)B(t) \tag{2.}$$

$$P_I(t,T) = E_t^{\mathbb{Q}}\left(\frac{P_I(t+\Delta,T)}{B(t+\Delta)}\right)B(t) = E_t^{\mathbb{Q}}\left(I(T)\cdot\frac{1_r}{B(T)}\right)B(t), \tag{3.}$$

where $E_t^{\mathbb{Q}}(\cdot) = E^{\mathbb{Q}}(\cdot\,|\mathscr{F}_t)$ is the time t conditional expectation using the martingale measure $\mathbb{Q}$.

These two expressions state that the current bond prices are equal to their expected discounted dollar cash flows at time T. The risk adjustment is captured via the martingale measure $\mathbb{Q}$ as distinct from the statistical measure $\mathbb{P}$.

4.3. Market Completeness

To discuss completeness, we first need to assume the market is arbitrage-free, so there exists an equivalent martingale measure $\mathbb{Q}$. Given $\mathbb{Q}$, the nominal and inflation-protected bond markets are complete with respect to $\mathbb{Q}$ (by definition) if (*a*) any payoff that is $\mathbb{Q}$ integrable and that depends on the evolution of these two term structures can be replicated synthetically by trading in the underlying nominal and inflation-protected zero-coupon bonds, including the nominal mma, and (*b*) the value process of the admissible self-financing trading strategy that replicates the payoff is a $\mathbb{Q}$ martingale. The second fundamental theorem of asset pricing states that the market is complete with respect to $\mathbb{Q}$ if and only if the martingale measure is unique (see Jarrow 2021, chapter 2). Sufficient conditions for the satisfaction of market completeness, once evolutions for the nominal and inflation protected bonds are given, are provided below.

There is some debate in the literature regarding whether the term structure of interest rate evolution has unspanned volatility, i.e., whether the interest rate market is complete due to stochastic volatility. The evidence is mixed (see, e.g., Bikbov & Chernov 2009; Collin-Dufresne, Goldstein & Jones 2009; Li & Zhao 2006). The problem with many of these studies rejecting the spanned volatility hypothesis is that they exogenously fix the number of factors in their estimation methodology (usually assuming there are up to three factors). Fixing the number of factors a priori can misspecify the model and bias the tests against accepting stochastic volatility and market completeness. If the market is incomplete, then unique valuation of interest rate and inflation derivatives using the methodology discussed next fails (see Jarrow 2021, chapter 8).

The successful use of these models in industry implies that the interest rate market being complete is a reasonable approximation. Consequently, we assume that the market is complete with respect to $\mathbb{Q}$ so that we can price various inflation derivatives. We note that we can price coupon-bearing bonds without assuming the market is complete because coupon-bearing bonds are linear combinations of the zero-coupon bonds. Completeness is needed for nonlinear functions of the zero-coupon bond price evolutions. Given the market is complete with respect to $\mathbb{Q}$, it can be shown that the cost of synthetically constructing an inflation derivative's payoffs is equal to its risk-neutral valuation, as given in the following formula.

Let X_T denote the payoff in nominal dollars to an inflation derivative at time T, where X_T is $\mathcal{F}_T$ measurable and $E^{\mathbb{Q}}\left(\frac{X_T}{B_T}\right) < \infty$, then the time $t \le T$ value of the inflation derivative

X_t satisfies

$$X_t = E_t^{\mathbb{Q}}\left(\frac{X_T}{B_T}\right)B_t. \tag{4}$$

This states that the present value of the inflation derivative's payoff at time $t \le T$ is the discounted expected value, where the adjustment for risk is captured in the use of the martingale probability $\mathbb{Q}$ and not $\mathbb{P}$. We will use this formula below when valuing various inflation derivatives.

For some applications, to compute explicit valuation formulas, a change of equivalent probability measures is useful. For this purpose, we define the forward price measure $\mathbb{Q}^T$ for $T \in [0, \mathscr{T}]$, first discovered by Jarrow (1987) and later independently again by Geman (1989), defined by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P_n(0,T)B_T} > 0. \tag{5}$$

This equivalent probability measure makes the nominal and inflation-adjusted zero-coupon bond prices, when normalized by the price of the nominal zero-coupon bond maturing at time T, $\mathbb{Q}^T$- martingales. That is,

$$\frac{P_n(t,\tau)}{P_n(t,T)}, \frac{P_I(t,\tau)}{P_n(t,T)}$$

for all $0 \le t \le \tau \le T$ are $\mathbb{Q}^T$- martingales.

Under this probability measure, the time t value of an inflation derivative payoff at some future date T can be written as

$$X_t = P_n(t,T)E_t^{\mathbb{Q}^T}(X_T). \tag{6}$$

This expression will be used in subsequent sections to simplify some valuation formulas.

4.4. Empirical Evolutions

For empirical implementation, evolutions for the term structure of nominal and inflation-adjusted zero-coupon bonds need to be specified. Three of the key models used in the literature and the industry are the Jarrow-Yildirim model (see Jarrow & Yildirim 2003), the affine model (see Ho, Huang & Yildirim 2014), and the market model of Brace, Gatarek & Musiela (2010) and Miltersen, Sandmann & Sondermann (1997). Although other models exist, we leave a discussion of their evolutions to the literature (see, e.g., Chen, Liu & Cheng 2010; Dam et al. 2020; Hinnerich 2008; Mercurio 2005; Singor et al. 2013). All of these models assume that the nominal and real bond markets are complete in order to use risk-neutral valuation to price the relevant inflation derivatives. Stochastic volatility term structure models are excluded from consideration because they imply an incomplete market where risk-neutral valuation fails (see Section 4.3 for additional discussion).

4.4.1. The HJM model. To present this model, we first need to define real and nominal forward rates. The nominal continuously compounded forward rate $f_n(t, T)$ at time t for maturity T (heuristically for the time period $[T, T + dt]$) is defined implicitly by

$$P_n(t,T) = e^{-\int_0^T f_n(t,s)ds}.$$

Similarly, the real continuously compounded forward rate $f_r(t, T)$ at time t for maturity T (heuristically for the future time period $[T, T + dt]$) is defined implicitly by

$$P_r(t,T) = e^{-\int_0^T f_r(t,s)ds}.$$

The assumed term structure evolutions are imposed on the forward rates and inflation index, as follows:

$$df_n(t,T) = \alpha_n(t,T)dt + \sum_{j=1}^{D} \sigma_{nj}(t,T)dW_j(t)$$

$$df_r(t,T) = \alpha_r(t,T)dt + \sum_{j=1}^{D} \sigma_{rj}(t,T)dW_j(t)$$

$$dI(t) = I(t)\left\{\mu(t)dt + \sum_{j=1}^{D} \sigma_{Ij}dW_j(t)\right\},$$

where $W_i(t)$ for $i = 1, \ldots, D$ are standard independent Brownian motions with $W_i(0) = 0$ for all i that generate the filtration $\mathbb{F}$, $[(\mu(t), \alpha_n(t,T), \alpha_r(t,T)]$ and $[\sigma_{ni}(t,T), \sigma_{ri}(t,T), \sigma_{Ii}(t,T)]$ for $i = 1, \ldots, D$ are $\mathbb{F}$ adapted and satisfy various regularity conditions so that the various stochastic integrals as implied by these evolutions are well defined.

These evolutions are very general, except for the implication that they generate term structure evolutions that have continuous sample paths, i.e., no jumps. This restriction, however, can also be easily relaxed (see Hinnerich 2008).

The no-arbitrage conditions on these evolutions are those that guarantee

$$\frac{P_n(t,T)}{B(t)}, \frac{P_I(t,T)}{B(t)},$$

for all $0 \leq t \leq T \leq \mathscr{T}$ are $\mathbb{Q}$ - martingales. These imply restrictions on the forward rates' and inflation rate's drifts under the martingale measures. The complete market conditions are those that guarantee the martingale measure is unique, which typically necessitate that the zero-coupon bonds' volatility matrices are nonsingular [e.g., for a specification of such conditions, see Amin & Jarrow (1991)].

For empirical applications, in order to obtain analytic expressions, two subcases of this general formulation are useful: the Jarrow-Yildirim model and the market model.

4.4.2. The Jarrow-Yildirim model. The Jarrow-Yildirim model is a special case of the HJM model obtained by letting the evolutions be given by

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n e^{v_n(T-t)}dW_n(t)$$

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r e^{v_r(T-t)}dW_r(t)$$

$$dI(t) = I(t)\left\{\mu(t)dt + \sigma_I dW_I(t)\right\},$$

where $\{\sigma_n, v_n, \sigma_r, v_r, \sigma_I\}$ are positive constants and $W_i(t)$ for $i \in \{n, r, I\}$ are correlated Brownian motions.

Under these restrictions, the forward rate processes are normally distributed and the inflation rate is lognormally distributed over any fixed time period under the equivalent martingale measure $\mathbb{Q}$, which facilitates analytic formulas for various IRD. The given evolutions imply the markets are complete.

The Jarrow-Yildirim model is a special case of an affine model for the evolution of the forward rate processes. For these augmented evolutions, see Ho, Huang & Yildirim (2014). For an extension that includes jumps, see Hinnerich (2008).

4.4.3. The market model. It is well-known that the market model is a special case of the HJM model (see Belgrade, Benhamou & Koehler 2005; Brace, Gatarek & Musiela 2010; Miltersen, Sandmann & Sondermann 1997). To obtain the market model, we need to define discrete forward rates at time t for the time interval $[T, T+\delta]$. These are defined by

$$1 + \delta F_n(t, T) = \frac{P_n(t, T)}{P_n(t, T+\delta)} \qquad 7.$$

$$1 + \delta F_r(t, T) = \frac{P_r(t, T)}{P_r(t, T+\delta)}. \qquad 8.$$

The right side of this expression isolates the implicit interest embedded in the different zero-coupon bonds over $[T, T+\delta]$. The discrete spot rates are $F_n(t, t)$ and $F_n(t, t)$ for $[t, t+\delta]$.

We also need to define the modified inflation index:

$$\mathcal{I}_{T+\delta}(t) = I(t)\frac{P_r(t, T+\delta)}{P_n(t, T+\delta)} = \frac{P_I(t, T+\delta)}{P_n(t, T+\delta)}.$$

The evolutions in this model are given under the forward rate measure $\mathbb{Q}^{T+\delta}$. The evolutions are assumed to satisfy

$$dF_n(t, T) = \sigma_n F_n(t, T) dW_n(t)$$

$$dF_r(t, T) = F_r(t, T)\left\{-\rho_{I,r}\sigma_I\sigma_r dt + \sigma_r dW_r(t)\right\}$$

$$d\mathcal{I}_{T+\delta}(t) = \sigma_I \mathcal{I}_{T+\delta}(t) dW_I(t),$$

where $W_i(t)$ for $i \in \{n, r, I\}$ are correlated Brownian motions under $\mathbb{Q}^{T+\delta}$, $\{\sigma_n, \sigma_r, \sigma_I\}$ are positive constants, and $\rho_{I,r}$ is the correlation between $W_r(t)$ and $W_I(t)$ under $\mathbb{Q}^{T+\delta}$.

Under these evolutions, the forward rate and modified inflation index processes are lognormally distributed under $\mathbb{Q}^{T+\delta}$, which facilitates analytic formulas for various inflation derivatives. We also note that under these evolutions,

$$\frac{P_n(t, \tau)}{P_n(t, T+\delta)}, \frac{P_I(t, \tau)}{P_n(t, T+\delta)}$$

for all $0 \leq t \leq \tau \leq T$ are $\mathbb{Q}^{T+\delta}$- martingales, which guarantee that the markets are arbitrage-free. The assumptions on the volatilities of these evolutions imply the markets are complete.

5. THE FISHER EQUATION AND THE BREAK-EVEN INFLATION RATE

This section uses the nominal and inflation-adjusted zero-coupon bonds to derive both the Fisher equation and the BEIR.

5.1. The Fisher Equation

The Fisher equation relates the nominal spot rate to the real spot rate and the expected inflation rate. To obtain this relation, we first rewrite Equation 3 as

$$P_I(t, T) = I(t)P_r(t, T) = E_t^{\mathbb{Q}}\left(\frac{I(t+\Delta)P_r(t+\Delta, T)}{B(t+\Delta)}\right)B(t). \qquad 9.$$

As seen, Equation 9 captures the relation among the inflation index, the nominal default-free interest rate, and the real default-free interest rates. Defining the inflation rate over $[t, t+\Delta]$ by

$\pi(t) = \frac{I(t+\Delta)}{I(t)} - 1$ and the discrete real spot rate of interest over $[t, t+\Delta]$ by $R_r(t) = \frac{1_r}{P_r(t,t+\Delta)} - 1$, we can derive the Fisher equation from Equation 9.

Theorem (Fisher equation). In an "arbitrage-free market" as discussed above, for small Δ,

$$r(t)\Delta \simeq R_r(t) + E_t^{\mathbb{P}}(\pi(t)) + \mathrm{Cov}_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}, \pi(t)\right) + R_r(t)E_t^{\mathbb{Q}}(\pi(t)). \qquad 10.$$

Proof. Using Equation 9, let $T = t + \Delta$. Then,

$1 = E_t^{\mathbb{Q}}\left(\frac{I(t+\Delta)}{I(t)} \frac{P_r(t+\Delta,t+\Delta)}{P_r(t,t+\Delta)} \frac{B(t)}{B(t+\Delta)}\right)$.

Define $R_n(t) = \frac{B(t+\Delta)}{B(t)} - 1$. Note $\frac{I(t+\Delta)}{I(t)} = 1 + \pi(t)$ and $\frac{P_r(t+\Delta,t+\Delta)}{P_r(t,t+\Delta)} = 1 + R_r(t)$ where $R_r(t)$ is nonrandom at time t.

When Δ is small, $R_n(t) \simeq r(t)\Delta$, which is nonrandom at time t.

Substitution yields

$1 + r(t)\Delta \simeq (1 + R_r(t))\, E_t^{\mathbb{Q}}(1 + \pi(t))$, or

$r(t)\Delta \simeq R_r(t) + E_t^{\mathbb{Q}}(\pi(t)) + R_r(t)E_t^{\mathbb{Q}}(\pi(t))$.

Next,

$E_t^{\mathbb{Q}}(\pi(t)) = E_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\pi(t)\right) = E_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right) E_t^{\mathbb{P}}(\pi(t)) + \mathrm{Cov}_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}, \pi(t)\right)$.

But, $E_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right) = 1$, which completes the proof. $\square$

The Fisher equation states that in an arbitrage-free market, the nominal default-free spot rate can be decomposed into the real rate of interest, the expected inflation rate under the statistical probabilities, an inflation risk premium, and an interaction term between the real and inflation rate.

5.2. The Break-Even Inflation Rate

The BEIR is best understood by considering the yields on the default-free nominal and inflation-adjusted zero-coupon bonds, as opposed to coupon bonds (see Section 6). As discussed in the next section, because one can always compute the zero-coupon bond prices implied by coupon bond prices, focusing on zero-coupon bond prices for this presentation is without loss of generality.

The BEIR at time t for the future time period T (heuristically for the future time period $[T, T+dt]$) is defined by

$$BEIR(t,T) = y_n(t,T) - y_r(t,T).$$

It is often incorrectly believed that this difference equals the expected inflation rate, but this is not the case, as the following theorem shows.

Theorem (BEIR decomposition).

$$\begin{aligned} BEIR(t,T) &= y_n(t,T) - y_r(t,T) \\ &= \frac{\log E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\right) + \log\left[1 + \dfrac{\mathrm{Cov}_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}, \frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right)}{E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\right)}\right]}{(T-t)} + \frac{1}{(T-t)}\log\left[1 + \dfrac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}\right]. \end{aligned} \qquad 11.$$

Proof. Using Equation 9, we have

$$\begin{aligned} \frac{P_r(t,T)}{1_r} &= E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)} \cdot \frac{1}{B(T)}\right) B(t) \\ &= E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right) E_t^{\mathbb{Q}}\left(\frac{1_n}{1_n B(T)}\right) B(t) + \mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right) \end{aligned}$$

$$= E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n} + E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}\frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}$$

$$= E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}\left[1 + \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}\right].$$

Taking logarithms yields

$$\log\left(\frac{P_r(t,T)}{1_r}\right) = \log E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right) + \log\left(\frac{P_n(t,T)}{1_n}\right) + \log\left[1 + \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}\right].$$

Substitution of the bond yields gives the following

$$-y_r(t,T)(T-t) = \log E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right) - y_n(t,T)(T-t) + \log\left[1 + \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}\right]$$

or

$$y_n(t,T) - y_r(t,T) = \frac{\log E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)}{(T-t)} + \frac{1}{(T-t)}\log\left[1 + \frac{\mathrm{Cov}_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}, \frac{B(t)}{B(T)}\right)}{E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)\frac{P_n(t,T)}{1_n}}\right].$$

Note that

$$E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right) = E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\cdot\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right)$$

$$= E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\right)E_t^{P}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right) + \mathrm{Cov}_t^{P}\left(\frac{I(T)}{I(t)}, \frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right).$$

But $E_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}\right) = 1$. Substitution into the previous expression gives the final result. □

This theorem states that the BEIR can be decomposed into three components. The first term, $\frac{\log E_t^{\mathbb{Q}}\left(\frac{I(T)}{I(t)}\right)}{(T-t)}$, is the continuous compounded expected inflation rate over the time period $[t, T]$. The second term is an inflation risk premium embedded in the BEIR. Finally, the third term is an adjustment necessary to account for the covariance between inflation and nominal spot rates over the time period $[t, T]$.

When $T = t + \Delta$, for small Δ, this reduces to the Fisher equation:

$$BEIR(t, t+\Delta) = r(t)\Delta - R_r(t)$$
$$\approx E_t^{\mathbb{P}}(\pi(t)) + \mathrm{Cov}_t^{\mathbb{P}}\left(\frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{\frac{d\mathbb{Q}}{d\mathbb{P}}(t)}, \pi(t)\right) + R_r(t)E_t^{\mathbb{Q}}(\pi(t)). \quad 12.$$

6. VALUING COUPON BONDS

In this section, we consider both nominal coupon bonds and inflation-adjusted coupon bonds, both of which are default-free. For both coupon bonds, simple valuation formula are obtained using just the assumption of no-arbitrage.

6.1. Nominal Coupon Bonds

A nominal coupon bond has a face value of $L \cdot 1_n$ dollars, a maturity time T, and a nominal dollar coupon $C \cdot 1_n$, paid at times $t = 1, 2, \ldots, T$. We let its time t value in nominal dollars be denoted $v_n(t, T: C, L)$.

It is well-known that a default-free coupon bond can be synthetically constructed as the portfolio of default-free zero-coupon bonds consisting of C zero-coupon bonds maturing at times $t = 1, 2, \ldots, T-1$ and $C+L$ zero-coupon bonds maturing at time T. Consequently, no-arbitrage implies that

$$v_n(t, T : C, L) = \sum_{j=[t]}^{T} CP_n(t, j) + (C+L)P_n(t, T), \tag{13.}$$

where $[t]$ denotes the smallest integer greater than or equal to t.

The yield on this coupon bond $Y_n(t, T: C, L)$ is the discrete rate[5] that, when used to discount the bond's cash flows, equates to the bond's price. This rate is implicitly defined by

$$v_n(t, T : C, L) = \sum_{j=[t]}^{T} \frac{C \cdot 1_n}{(1 + Y_n(t, T : C, L))^{j-t}} + \frac{(C+L) \cdot 1_n}{(1 + Y_n(t, T : C, L))^{T-t}}. \tag{14.}$$

As defined, the bond's yield is known at time t.

6.2. Inflation-Adjusted Coupon Bonds

An inflation-adjusted coupon bond has a maturity of T, a notional value of $L \cdot 1_r$ real dollars, and a coupon of $C \cdot 1_r$ real dollars at times $t = 1, 2, \ldots, T$. But the payoffs are in dollars: $I(T)L \cdot 1_r$ at time T and $I(t)C \cdot 1_r$ at times $t = 1, \ldots, T$. Note that the dollar payoffs increase by the inflation index. We let the time t value in nominal dollars be denoted $v_I(t, T : C, L)$.[6]

It can be shown that a default-free inflation-adjusted coupon bond can be synthetically constructed as the portfolio of default-free inflation-adjusted zero-coupon bonds consisting of C inflation-adjusted zero-coupon bonds maturing at times $t = 1, 2, \ldots, T-1$ and $C+L$ inflation-adjusted zero-coupon bonds maturing at time T. Consequently, no-arbitrage implies that

$$v_I(t, T : C, L) = \sum_{j=[t]}^{T} CP_I(t, j) + (C+L)P_I(t, T). \tag{15.}$$

The yield on an inflation-adjusted coupon bond $Y_I(t, T : C, L)$ is implicitly defined by

$$v_I(t, T : C, L) = \sum_{j=[t]}^{T} \frac{I(t)C \cdot 1_r}{(1 + Y_I(t, T : C, L))^{j-t}} + \frac{I(t)(C+L) \cdot 1_r}{(1 + Y_I(t, T : C, L))^{T-t}}. \tag{16.}$$

We can use Equation 1 to rewrite the inflation-adjusted coupon bond's price using the hypothetical real zero-coupon bonds as:

$$v_I(t, T : C, L) = \sum_{j=[t]}^{T} CI(t)P_r(t, j) + (C+L)I(t)P_r(t, T). \tag{17.}$$

---

[5]To be consistent with the zero-coupon bond's yield, we could have defined the coupon bond's continuously compounded yield. Since we will not use the coupon bond's yield in subsequent computations, we choose to make the definition more close to that used in the industry when quoting market prices.

[6]In the United States, inflation-adjusted bonds include an embedded put option that guarantees that the principal is never less than par. The valuation of such put options is discussed in a subsequent section.

Defining a hypothetical coupon bond paying in real dollars with a maturity of T, a notional value of $L \cdot 1_r$ real dollars, a coupon of $C \cdot 1_r$ real dollars at times $t = 1, 2, \ldots, T$, and a time t price denoted $v_r(t, T : C, L)$, we have that

$$v_I(t, T : C, L) = I(t)v_r(t, T : C, L), \tag{18.}$$

where

$$v_r(t, T : C, L) = \sum_{j=[t]}^{T} CP_r(t, j) + (C + L)P_r(t, T). \tag{19.}$$

6.3. Stripping Coupon Bond Prices

In most markets, zero-coupon nominal and inflation-adjusted bonds only trade for a few short maturities, and the remaining maturities are coupon-bearing bonds. In such a situation, to obtain the prices of the underlying nominal zero-coupon bonds $P_n(t, T)$ one can use Equation 13, given various coupon bonds $\{(T : C, L)\}$ to obtain a system of linear equations in unknowns, whose solution gives the nominal zero-coupon bond prices for a collection of maturities $t \leq T \leq \mathscr{T}$. The collection of zero-coupon bond prices obtained correspond to the coupon payment dates underlying the collection of coupon bonds used in the estimation. Similarly, Equation 15 can be used to obtain the prices of the inflation-adjusted zero-coupon bond prices $P_I(t, T)$ for a collection of maturities $t \leq T \leq \mathscr{T}$. And finally, Equation 1 and knowledge of the inflation index generates the real zero-coupon bond prices $P_r(t, T)$. This method is called stripping the coupon bond prices.

As noted, however, the procedure just described only gives a finite collection of zero-coupon bond prices corresponding to the coupon payment dates underlying the collection of coupon bonds used in the estimation. To price various inflation derivatives, it is desired to obtain the implied zero-coupon bond prices for the entire continuum of maturities between time t and $\mathscr{T}$. This procedure is called term structure smoothing. The techniques for smoothing the nominal term structure of interest rates are well-known, and these techniques can be employed to obtain both the inflation-adjusted and real zero-coupon bond price curves. We leave a discussion of these smoothing techniques to the existing literature (for a review, see Jarrow 2014).

7. INFLATION SWAPS

There are two types of popular inflation swaps: zero-coupon and year-on-year payment swaps. This section discusses both types.

7.1. Zero-Coupon Inflation Swaps

For simplicity, we assume the inflation swap has a notional value (assumed to be 1_n) and a maturity date T. The swap has two legs: fixed and floating. We consider a swap written at time 0.

1. The fixed leg pays a fixed and constant per period nominal interest rate C times the dollar notional value 1_n at time T, compounded discretely over the swap's life (from 0 to T). In symbols, this equals
$$\left((1 + C)^T - 1\right) 1_n.$$
2. The floating legs receive in dollars the inflation rate over $[0, T]$, i.e.,
$$\left(\frac{I(T)}{I(0)} - 1\right) 1_n.$$

No cash changes hands when the swap is initiated. Hence, it has zero value at time 0. The net cash flow to the zero-coupon inflation swap at time T is

$$\left(\frac{I(T)1_r}{I(0)1_r}\right)1_n - (1+C)^T 1_n = \left(\frac{P_I(T,T)}{I(0)1_r}\right)1_n - (1+C)^T 1_n.$$

The inflation swap rate is that C that makes the swap have zero value at initiation.

This swap's payoff can be synthetically constructed by buying $\frac{1_n}{I(0)1_r} = 1$ unit of the inflation-adjusted zero-coupon bond maturing at time T and shorting $(1+C)^T$ units of a nominal zero-coupon bond maturity at time T. Indeed, this buy and hold portfolio has the same payoffs as the swap.

Hence, no-arbitrage implies that the value of a zero-coupon bond inflation swap at some time $t \in [0, T]$ denoted $S_z(t, T : C)$ is

$$S_z(t, T : C) = \left(\frac{1_n}{I(0)1_r}\right)P_I(0,T) - (1+C)^T P_n(0,T).$$

The zero-coupon inflation swap rate at time 0 is that C such that the swap has zero, i.e.,

$$C = \left[\frac{P_I(0,T)}{P_n(0,T)} \cdot \frac{1_n}{I(0)1_r}\right]^{\frac{1}{T}} - 1$$

or, in terms of yields,

$$C = e^{\frac{y_n(0,T)-y_r(0,T)}{T}} - 1 = e^{\frac{BEIR(0,T)}{T}} - 1.$$

This shows that the continuously compounded equivalent zero-coupon inflation swap rate equals the BEIR, i.e., $\frac{ln(1+C)}{T} = BEIR(0,T)$.

7.2. Year-On-Year Inflation Swaps

This section values year-on-year inflation swaps in two ways: first, without assuming a particular evolution for the nominal and inflation bond term structure; and second, assuming a particular evolution process.

7.2.1. Evolution-free valuation. A year-on-year inflation swap has a notional value (assumed to be 1_n), a maturity date T, and a payment frequency from $t = 1, \ldots, T$.

The swap has two legs: fixed and floating. We consider a swap written at time 0.

1. The fixed leg pays a fixed and constant nominal interest rate C times the notional value at each payment date. In symbols, this equals

$$C1_n.$$

2. The floating leg receives the inflation rate over each intermediate period $[t-1, t]$ times the dollar notional value, i.e., $\left(\frac{I(t)}{I(t-1)} - 1\right)1_n$.

No cash changes hands when the swap is initiated. Hence, it has zero value at time 0. The net cash flow to the zero-coupon inflation swap at each time t is

$$\left(\frac{I(t)}{I(t-1)} - 1\right)1_n - C1_n.$$

The inflation swap rate is that C that makes the swap have zero value at time 0.

The time t dollar value of the swap for an arbitrary C is denoted $S_y(t, T : C)$. Since the market is complete, we can use risk-neutral valuation to value this swap. Its value is

$$S_y(t,T:C) = E_t^{\mathbb{Q}}\left[\sum_{j=[t]}^{T}\left(\frac{\frac{I(j)}{I(j-1)}-1}{B(j)}\right)1_n - \sum_{j=[t]}^{T} C\frac{1_n}{B(j)}\right]B(t).$$

This can alternatively be written as:

$$S_y(t,T:C) = \sum_{j=[t]}^{T} E_t^{\mathbb{Q}}\left(\frac{1_n}{I(j-1)1_r}\cdot\frac{P_I(j-1,j)}{B(j-1)}\right)B(t) - \sum_{j=[t]}^{T}(1+C)P_n(t,j).$$

Proof. Given,

$$\begin{aligned}
S_y(t,T:C) &= E_t^{\mathbb{Q}}\left[\sum_{j=[t]}^{T}\left(\frac{\frac{I(j)}{I(j-1)}-1}{B(j)}\right)1_n - \sum_{j=[t]}^{T} C\frac{1_n}{B(j)}\right]B(t) \\
&= E_t^{\mathbb{Q}}\left[\sum_{j=[t]}^{T}\left(E_{j-1}^{\mathbb{Q}}\left(\frac{\frac{I(j)}{I(j-1)}-1}{B(j)}\right)1_n\frac{B(j-1)}{B(j-1)}\right)B(t)\right] - \sum_{j=[t]}^{T} CE_j^{\mathbb{Q}}\left[\frac{1_n}{B(j)}\right]B(t) \\
&= E_t^{\mathbb{Q}}\left[\sum_{j=[t]}^{T}\left(\frac{1_n}{I(j-1)1_r}E_{j-1}^{\mathbb{Q}}\left(\frac{I(j)1_r}{B(j)}\right)\frac{B(j-1)}{B(j-1)}B(t)\right)\right] - \sum_{j=[t]}^{T} E_j^{\mathbb{Q}}\left[\frac{1_n}{B(j)}\right]B(t) \\
&\quad - \sum_{j=[t]}^{T} CE_j^{\mathbb{Q}}\left[\frac{1_n}{B(j)}\right]B(t) \\
&= \sum_{j=[t]}^{T} E_t^{\mathbb{Q}}\left(\frac{1_n}{I(j-1)1_r}\cdot\frac{P_i(j-1,j)}{B(j-1)}\right)B(t) - \sum_{j=[t]}^{T}(1+C)P_n(t,j). \qquad \square
\end{aligned}$$

This equation cannot be simplified further without an evolution due to the first term.

The inflation swap rate at time 0 is that nominal rate C is such that the swap has zero value, i.e.,

$$C = \frac{\sum_{j=1}^{T} E_0^{\mathbb{Q}}\left(\frac{1_n}{I(j-1)1_r}\cdot\frac{P_I(j-1,j)}{B(j-1)}\right)B(t) - \sum_{j=1}^{T} P_n(0,j)}{\sum_{j=1}^{T} P_n(0,j)}.$$

7.2.2. Empirical models. Under either the Jarrow-Yildirim, affine, or market models, analytic expressions for the time t value of intermediate inflation swaps are known (Ho, Huang & Yildirim 2014; Mercurio 2005). For extensions of these models to include jumps, see Hinnerich (2008).

8. ARBITRAGE: BONDS AND INFLATION SWAPS

This section shows how to construct a default-free nominal coupon bond using a default-free inflation-adjusted coupon bond and a collection of zero-coupon inflation swaps.

8.1. The Synthetic Construction

Given the simultaneous trading in default-free inflation-adjusted bonds and inflation swaps, one can synthetically construct a synthetic default-free nominal bond in the following fashion. First, since a coupon bond is a portfolio of zero-coupon bonds (see Section 6), if we can synthetically construct a default-free zero-coupon nominal bond with a default-free inflation-adjusted bond and an inflation swap, then by combining these in a portfolio we can construct the coupon bond. Hence, we show how to construct a default-free zero-coupon nominal bond with a default-free inflation-adjusted bond and a zero-coupon inflation swap.

At time 0, consider holding an inflation-adjusted zero-coupon bond maturing at time T. Its payoff is

$$I(T) \cdot 1_r$$

at time T. Suppose one also shorts $I(0)1_r = 1$ unit of the zero-coupon inflation swap with a notional value of 1_n at time T. The time T payoff of the zero-coupon inflation swap is

$$-I(0)1_r \left[\left(\frac{I(T)1_r}{I(0)1_r} \right) 1_n - (1+C)^T 1_n \right] = -I(T)1_r + (1+C)^T 1_n.$$

The combined portfolio's time T payoff is therefore

$$I(T)1_r - I(T)1_r + (1+C)^T 1_n = (1+C)^T 1_n.$$

This corresponds to the payoff from holding $(1+C)^T$ units of a default-free nominal zero-coupon bond maturing at time T. To get a single unit of the default-free nominal zero-coupon bond maturing at time T, just reduce the initial portfolio's holdings by the fraction $\frac{1}{(1+C)^T}$. This completes the argument.

8.2. The Empirical Evidence

The above construction implies that in an arbitrage-free, frictionless, and competitive market the synthetic default-free nominal coupon bond should cost the same to construct as the traded default-free nominal coupon bond. Fleckenstein, Longstaff & Lustig (2014) investigated this equivalence using US Treasury securities from July 2004 and November 2009. They found that the two prices were significantly different and explored which market frictions could explain the difference. They concluded that they could not really explain the difference using the standard market frictions.

9. OPTIONS

This section studies inflation-indexed caps and floors, inflation-indexed swaptions, inflation spread options, and inflation-linked equity options. We provide valuation formulas for all these IRD.

9.1. Inflation-Indexed Caps and Floors

An inflation-indexed cap is an inflation derivative that has a set of payment dates denoted $t \in \{\delta, 2\delta, \ldots, 1, \ldots, \tau - \delta, \tau\}$, where δ corresponds to a fraction of a year, say 1/4, or 3 months. The cap's start date is time 0. The maturity date of the cap is τ, and its notional value is 1_n. At each payment date, the cap's payoff corresponds to the payoff of an inflation-indexed caplet whose maturity corresponds to the payment date. Each caplet has the same strike rate κ, the same notional value 1_n, and the same underlying inflation index $I(t)$.

In particular, one of these inflation-indexed caplets is a European call option on the inflation index. It has a maturity date $T \in \{\delta, 2\delta, \ldots, 1, \ldots, \tau - \delta, \tau\}$, notional value 1_n, and a strike rate κ. It is written on the inflation rate realized over the time period $[T - \delta, T]$, i.e., $\left(\frac{I(T)}{I(T-\delta)} - 1\right)$. The time T payoff in nominal dollars to such an inflation-indexed caplet is

$$\max\left[\left(\frac{I(T)}{I(T-\delta)}-1\right)-\kappa,0\right]1_n\delta.$$

As seen, the caplet earns the difference between the inflation rate over the time period $[T-\delta, T]$ and the strike rate κ times the notional value 1_n and is prorated by δ, but only if this difference is strictly positive.

Its time t arbitrage-free value, denoted $c(t, T\colon \kappa, \delta)$, is given by

$$c(t,T:\kappa,\delta)=E_t^{\mathbb{Q}}\left[\frac{\max\left[\left(\frac{I(T)}{I(T-\delta)}-1\right)-\kappa,0\right]1_n\delta}{B(T)}\right]B(t),$$

using the equivalent martingale probability $\mathbb{Q}$, or by

$$c(t,T:\kappa,\delta)=P_n(t,T)E_t^{\mathbb{Q}^T}\left[\max\left[\left(\frac{I(T)}{I(T-\delta)}-1\right)-\kappa,0\right]1_n\delta\right],$$

using the forward martingale probability $\mathbb{Q}^T$. The arbitrage-free value of the cap is just the sum of all the values of the caplets from which it is composed.

An inflation-indexed floor with the same characteristics $\{1_n, \tau, \kappa, \delta\}$ is defined similarly, except that its payoffs correspond to those from a collection of inflation-indexed floorlets. And, a floorlet is just a European put option on the inflation index. Given the floorlet's characteristics $\{1_n, T, \kappa, \delta\}$ for $T \in \{\delta, 2\delta, \ldots, 1, \ldots, \tau-\delta, \tau\}$, its payoffs are given by

$$\max\left[\kappa-\left(\frac{I(T)}{I(T-\delta)}-1\right),0\right]1_n\delta.$$

Similarly, arbitrage-free valuation formulas hold as for the caplets. Given specific evolutions as in Section 4.4, explicit analytic formulas are available for caps and floors (Hinnerich 2008; Ho, Huang & Yildirim 2014; Mercurio 2005).

9.2. Inflation-Indexed Swaptions

An inflation-indexed swaption is an option with a fixed maturity date to enter into a particular inflation swap, or alternatively stated, it is a European call option with a strike price of zero and a fixed maturity date on a particular inflation swap, where the maturity of the swap exceeds that of the option. There are two types of swaptions: those written on zero-coupon inflation swaps and those written on year-on-year inflation swaps. We consider both of these in this section.

9.2.1. Swaptions on zero-coupon inflation swaps. Consider the zero-coupon inflation swap with maturity T and swap rate C with time $t \in [0, T]$ value denoted $S_z(t, T : C)$. This swap was discussed in Section 7.1 above.

The inflation swaption on this zero-coupon inflation swap is a European option with maturity $\tau < T$ and a strike price of zero, with time τ payoff

$$\max\left[S_z(\tau,T:C),0\right].$$

The time t arbitrage-free value, denoted $swap_z(t, T)$, is

$$swap_z(t,T)=E_t^{\mathbb{Q}}\left[\frac{\max\left[S_z(\tau,T:C),0\right]}{B(T)}\right]B(t).$$

9.2.2. Swaptions on year-on-year inflation swaps.

Consider the year-on-year inflation swap with maturity T and swap rate C with time $t \in [0, T]$ value denoted $S_y(t, T : C)$. This swap was discussed in Section 7.2 above.

The inflation swaption on this year-on-year inflation swap is a European option with maturity $\tau < T$ and a strike price of zero, with time τ payoff

$$\max\left[S_y(\tau, T : C), 0\right].$$

The time t arbitrage-free value, denoted $swap_y(t, T)$, is

$$swap_y(t, T) = E_t^{\mathbb{Q}}\left[\frac{\max\left[S_z(\tau, T : C), 0\right]}{B(T)}\right]B(t).$$

9.2.3. Closed-form solutions.

Closed-form solutions for these various inflation-indexed swaptions under particular evolutions can be found in Ho, Huang & Yildirim (2014) and Hinnerich (2008).

9.3. Inflation Spread Options and Hybrid Products

Various other inflation derivatives trade over-the-counter. Two examples are inflation spread options and a hybrid product called an inflation-linked equity option (see Kerkhof 2005).

An inflation spread option pays off, in nominal dollars, the difference between two inflation indices, which often are in different countries. We have the inflation index in dollars, $I(t)$, as defined earlier. Consider an alternative index $\mathbb{I}(t)$ on the inflation rate in another country. A spread option on these two rates, created at time 0 with maturity T and time $t \in [0, T]$ value denoted $O_{\mathbb{I}}(t, T)$, has a time T payoff equal to

$$\max\left[\frac{I(T)}{I(0)} - \frac{\mathbb{I}(T)}{\mathbb{I}(0)}, 0\right]1_n.$$

The spread option only pays off if the inflation rate spread is strictly positive over the time interval $[0, T]$. If the market is arbitrage-free and enough securities trade so that the market is complete, considering the new inflation index $\mathbb{I}(t)$, then the time t value of this option satisfies

$$O_{\mathbb{I}}(t, T) = E_t^{\mathbb{Q}}\left[\frac{\max\left[\frac{I(T)}{I(0)} - \frac{\mathbb{I}(T)}{\mathbb{I}(0)}, 0\right]1_n}{B(T)}\right]B(t).$$

To consider an inflation-linked equity option, let $S(t)$ denote the time t value of an equity or equity index. An inflation-linked equity option, created at time 0 with maturity T and time $t \in [0, T]$ value denoted $O_S(t, T)$, has a time T payoff equal to

$$\max\left[I(T)1_r - S(T), 0\right].$$

The option pays off the difference between the value of two assets: the inflation-protected default-free bond $P_I(t, T)$ and the equity $S(t)$. Again, if the market is arbitrage-free and enough securities trade so that the market is complete, then the time t value of this option satisfies

$$O_S(t, T) = E_t^{\mathbb{Q}}\left[\frac{\max\left[I(T)1_r - S(T), 0\right]}{B(T)}\right]B(t).$$

10. THE EMPIRICAL LITERATURE

The empirical literature can be divided into two streams: (*a*) estimating expected inflation rates or, equivalently, inflation risk premiums, and (*b*) a comparison of the various inflation derivative valuation models and the efficiency of the inflation derivatives market.

10.1. Expected Inflation Rates

The empirical literature on inflation-adjusted bonds has primarily focused on using the Fisher equation (Equation 10) or, equivalently, the BEIR equation (Equation 11) to obtain estimates of the expected inflation rate $\left(\frac{\log E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\right)}{(T-t)}\right)$ and the inflation risk premium $\left(\log\left[1+\frac{\mathrm{Cov}_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)},\frac{d\mathbb{Q}}{d\mathbb{P}}(t)\right)}{E_t^{\mathbb{P}}\left(\frac{I(T)}{I(t)}\right)}\right]\right)$ for various future time horizons $[t, T]$.

Estimates of these expected inflation rates are useful for guiding monetary policy, and estimates of the risk premium are beneficial to governments in deciding whether to finance government deficits by issuing either nominal or inflation-adjusted bonds. For example, if inflation risk premiums are positive, then issuing inflation-adjusted bonds that reflect no inflation risk premium provides a cheaper borrowing instrument than nominal bonds. In addition, if there is a positive inflation risk premium, then investors' welfares are improved by issuing inflation-adjusted bonds because they provide otherwise unavailable hedging instruments (i.e., it completes the market with respect to inflation risk).

To decompose the BEIR into its various components using either the term structures of nominal and inflation-adjusted bonds yields or zero-coupon inflation swap rates, one must perform a number of steps.

1. Select a database for bond prices, including the observation interval (daily, weekly, monthly).
2. Estimate the real and nominal term structures from the database, using a forward rate smoothing procedure to obtain yields or forward rates of various maturities (see Section 6.3).
3. Assume an evolution for the term structures (see Section 4.4), including a process for the market price of risk $\frac{d\mathbb{Q}}{d\mathbb{P}}(t)$ perhaps via an equilibrium model, and estimate the parameters of these processes.
4. Adjust for possible liquidity risk premiums, given the different liquidities between the traded nominal and inflation-adjusted bonds in most countries.

Because of the plethora of choices in the implementation of steps 1–4, there is no consensus in the literature as to how expected inflation rates or risk premiums should be estimated. These differences have led to a wide variation in the inflation risk premium estimates obtained in the literature from positive to negative over the same time periods (e.g., see Ang, Bekaert & Wei 2008; Bekaert & Wang 2010; Buraschi & Jiltsov 2005; Chen, Engstrom & Grishchenko 2016; Christensen, Lopez & Rudebusch 2010; Kaminska et al. 2018). In a recent paper, Chipeniuk & Walker (2021) used inflation cap and floors data to estimate inflation expectations. For a comprehensive review of this literature, see Kupfer (2018). Given the lack of consensus in the empirical asset pricing literature on how to estimate equity premiums (see Jagannathan, Schaumburg & Zhou 2010), this absence of a consensus for estimating inflation risk premiums is easy to understand.

10.2. Efficiency of the Inflation Derivatives Market

Unlike the empirical literature on the efficiency of equity option markets and a comparison of the various valuation models (see Bates 2022), the empirical literature on the valuation of inflation derivatives is almost exclusively represented by a collection of theoretical models, supported by illustrations of how to compute their values (see Chen, Liu & Cheng 2010; Dam et al. 2020; Hinnerich 2008; Ho, Huang & Yildirim 2014; Jarrow & Yildirim 2003; Mercurio 2005; Singor et al. 2013). An exception to this is the paper by Fleckenstein, Longstaff & Lustig (2014), which studies a TIPS market inefficiency, as discussed in Section 8 above. A comprehensive empirical investigation of the efficiency of the inflation derivatives markets and a comparison of the various available models are open research questions yet to be explored.

11. CONCLUSION

This article reviews the literature on inflation-adjusted bonds, swaps, and derivatives. As documented, the theoretical literature is quite mature, utilizing the existing theories developed in the term structure literature. Empirically, to date, the application of these insights has mainly focused on computing expected inflation rates and inflation risk premiums to help guide monetary policy decision-making. The applications of these insights to the valuation and hedging of inflation derivatives and to understanding the efficiency of the inflation derivative markets are open research questions, yet to be investigated.

DISCLOSURE STATEMENT

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

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