David Benney circa 1991,
when he was Head of the MIT Mathematics Department

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David J. Benney: Nonlinear Wave and Instability Processes in Fluid Flows

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Abstract

David J. Benney (1930–2015) was an applied mathematician and fluid dynamicist whose highly original work has shaped our understanding of nonlinear wave and instability processes in fluid flows. This article discusses the new paradigm he pioneered in the study of nonlinear phenomena, which transcends fluid mechanics, and it highlights the common threads of his research contributions, namely, resonant nonlinear wave interactions; the derivation of nonlinear evolution equations, including the celebrated nonlinear Schrödinger equation for modulated wave trains; and the significance of three-dimensional disturbances in shear flow instability and transition.

1. INTRODUCTION

1.1. From New Zealand to MIT

David John Benney (April 8, 1930–October 9, 2015) was an applied mathematician and fluid dynamicist. He was born in Wellington, New Zealand, and started his education in a one-room schoolhouse near the town of Te Awamutu. He went on to Victoria University College (a branch of what was then the New Zealand University), where he was originally going to study either French and German or geology, but he was soon turned on to mathematics, receiving his Bachelor of Science (B.Sc.) with first-class honors in 1950 and a Masters of Science (M.Sc.) in 1951 (**Figure 1**). He continued his education at Emmanuel College, University of Cambridge, in England on a New Zealand postgraduate scholarship. Benney excelled in Parts II and III of the Mathematical Tripos, receiving a B.A. in mathematics with first-class honors in 1954, but he did not stay for a PhD in Cambridge. Instead, following a three-year (1954–57) stint as Lecturer at Canterbury University College back in New Zealand, he entered the applied mathematics doctoral program at MIT (Massachusetts Institute of Technology) in Cambridge, Massachusetts, and completed his PhD in 1959 in record time. Just two days after graduation, he married Elizabeth (Liz) Benney (née Matthews), a fellow New Zealander he had met on a ski trip a few years earlier. They had

Figure 1

David J. Benney circa 1950 when he was a student at Victoria University College. Image courtesy of Elizabeth Benney.

three children: Richard, Paul, and Antonia. David Benney spent his entire career on the faculty in the mathematics department at MIT, rising from Instructor in 1959 to Full Professor in 1966 and Department Head (1989–99), until his retirement in 2010.

In the 1950s, applied mathematics at MIT was led by Chia-Chiao (C.C.) Lin, a former student of Theodore von Kármán at Caltech and recognized leader in theoretical fluid dynamics, particularly hydrodynamic stability, and by Eric Reissner, a renowned expert in elasticity theory. Reissner was also the managing editor of the *Journal of Mathematics and Physics*, one of the oldest journals in the United States that specialized in the interplay between mathematics and physical applications, founded by Clarence Moore at MIT in 1921. Although he came to MIT to work on hydrodynamic stability on the recommendation of George Batchelor at Cambridge University, who knew C.C. Lin, Benney toyed with the idea of working on elasticity after attending an excellent course on partial differential equations taught by Reissner, but he finally decided to stay with the original plan. His doctoral thesis, under the supervision of C.C. Lin, drew attention to the role of nonlinear interactions between two- and three-dimensional (3D) disturbances in initiating transition in parallel shear flows, as observed in experiments for a boundary layer at the National Bureau of Standards. This approach was a clear departure from the classical, linear theory of shear flow instability based on the Orr–Sommerfeld equation, and it also set the stage for Benney's later seminal contributions on resonant nonlinear wave interactions. Furthermore, the use of perturbation expansions for analyzing interacting shear flow instability modes in his PhD thesis was a precursor of the multiple-scale asymptotic techniques pioneered by Benney in later studies.

The late 1950s through 1960s were a period of expansion and reorganization of applied mathematics at MIT. In the area of fluid mechanics, in particular, apart from Benney, several new appointments were made, including Louis N. Howard, Harvey P. Greenspan, Steven A. Orszag, and Willem V.R. Malkus. This increased interest in fluid mechanics with an applied mathematics flavor was spurred in part by the parallel development of geophysical fluid dynamics (GFD), which brought to center stage the need for theoretical models of fluid phenomena controlled by the effects of stratification, rotation, and shear. Several advances in this new field were initiated at the GFD Program held every summer since 1959 at Woods Hole, Massachusetts, with strong MIT participation. Although Benney was not directly involved in this program, starting in the mid-1960s, his research interests in nonlinear internal gravity waves and stratified shear flows were clearly influenced by GFD applications. In 1968 David Benney became Managing Editor of the *Journal of Mathematics and Physics*, which was renamed *Studies in Applied Mathematics*, and its scope was focused on publishing articles originating from or invited by the MIT applied mathematics group. Under his stewardship for the next 46 years, *Studies* was established as one of the leading journals in physical applied mathematics.

1.2. Research Paradigm

Overall, Benney's theoretical approach is typical of the British school of applied mathematics in so far as physical motivation plays an important part in the choice and solution of problems; rather than detailed studies of very specific flows, however, Benney's focus was on developing generic theoretical models that capture the underlying physics of nonlinear phenomena in a variety of physical contexts. This research paradigm has proved particularly fruitful, and the impact of his work transcends fluid mechanics. For example, Benney and his first PhD student, Alan Newell, derived in a systematic way the celebrated nonlinear Schrödinger (NLS) equation (Benney & Newell 1967b) as the canonical evolution equation of the envelope of a weakly nonlinear modulated wave packet. Like many canonical equations, the NLS equation was also discovered at almost the same time in the USSR, by Vladimir E. Zakharov and coworkers (e.g., Zakharov 1968). This is a fundamental result in fluid mechanics that also finds applications in far-removed areas such as nonlinear optics, plasma physics, and Bose–Einstein condensates. Furthermore, Benney (1966a) pointed out that the Korteweg–de Vries (KdV) equation, previously familiar in the context of shallow-water waves, governs weakly nonlinear long-crested waves in a fluid layer under general flow conditions—with or without a free surface or in the presence of density stratification, shear, and rotation—a contribution that in addition to fluid mechanics has had lasting impact in meteorology and oceanography.

Being a physical applied mathematician, Benney appreciated pure mathematics but had little interest in pursuing abstract mathematical analysis. Nonetheless, some of the model equations he derived in the context of fluid mechanics have attracted considerable attention from the pure mathematics community. An example is the so-called Benney System (Benney 1973), which generalizes the classical shallow-water equations to flows with nonzero vorticity and turns out to have interesting mathematical properties, including an infinite number of conservation laws.

1.3. Style and Idiosyncrasies

Benney's papers are typically short and to the point: Following a brief introduction that touches on the motivation for the paper and its relation to earlier work, the main results are presented with only a few key steps of the usually extensive algebraic manipulations. While this minimalistic style does not make for easy reading, it reflects David Benney's emphasis on truly new ideas devoid of embellishment: "Dave can never be accused of writing two words where one will do," noted Peter J. Bryant (Bryant 1990, p. 2), who was a student at Canterbury University College when David taught there. However, economy of presentation did not seem to deter readers, as his two most cited papers (Benney 1966b, Benney & Newell 1967b) are only six pages long and the introduction is limited to a single paragraph; the novelty of the ideas put forward in these relatively short papers greatly outweighed the brevity of the discussion.

David Benney had a generally understated presence and was very much at ease with himself. His quick wit and subtle sense of humor (which sometimes was only appreciated after one had left his office) were most characteristic of his personality. While occasionally he may have appeared brusque to some, his overall fairness, consideration, and respect for colleagues and students were undeniable. These attributes along with a straightforward, no-nonsense leadership style were most appreciated during his ten-year tenure as Department Head (1989–99).

As an educator, Benney took classroom teaching very seriously both at the undergraduate level (he coauthored with H.P. Greenspan the text *Calculus: An Introduction to Applied Mathematics*) and the graduate level, where he developed advanced subjects on perturbation asymptotics and nonlinear wave motion. He was also a dedicated mentor of graduate students and continued to be supportive of them even after graduation. He supervised 18 PhD students, several of whom went on to have distinguished careers in academia.

Throughout his career, David Benney cultivated a low profile, not seeking recognition and even refusing to accept well-deserved honors or awards. His aversion to self-promotion dates back to the time he was a student, as illustrated by the following story told by his wife, Liz. While at Victoria University College, David was recommended to apply for a Rhodes Scholarship for postgraduate studies at the University of Oxford, and as part of the application, he had to be interviewed by the Governor General of New Zealand. At that time, it so happened that David had a summer job helping in the gardens of the Government House, so he saw the Governor General almost daily when they discussed the growing vegetables and fruit. On the day of the interview, after finishing his job, David went home to change clothes and returned to the Government House for the interview. The Governor General at first did not recognize his young gardener dressed in a jacket and tie, but as the interview progressed he said to David, "Somehow I think I know you." However, David, not wanting to push himself, did not reveal that he worked in the Governor's gardens!

About 50 years later, in 2000, to celebrate David's 70th birthday, his former PhD students decided to organize in his honor a conference at MIT with invited speakers and a banquet. However, in view of his allergy to any sort of recognition, they faced a major challenge: How to convince the honoree to attend? To overcome this obstacle, they sought the help of Liz. She agreed to keep the celebration secret from Dave, and on the day of the banquet, which preceded the conference, she managed to bring him to the banquet venue with the excuse that they were going to attend the presentation of a new book by a friend. The plot worked perfectly: Although it took several minutes for Dave to get over the initial shock, the banquet was a big success and it was clear that he was extremely happy and grateful for this honor. The conference presentations along with a brief summary of Dave's research contributions (Ablowitz et al. 2002) were published in a special issue of *Studies in Applied Mathematics*.

1.4. Farmer Benney

David Benney had an interest in the outdoors and sports since his childhood in New Zealand. He was an avid skier and hiker and an excellent tennis player. In the last twenty years of his life, however, he devoted most of his free time to farming. This new hobby/occupation came about as the Benneys made a decision to leave Wayland, Massachusetts, where their children grew up, and buy a farm. Liz had been passionate about horses since she was a small child in New Zealand and she became a top rider at a young age. In the United States, she participated as both a competitor and a judge at equestrian events, winning many awards, but her dream was to own and run a real working farm. This lifelong vision was realized in 1989 when the Benneys bought seventy-plus acres of land with an old farmhouse and two antique barns in the town of Upton, Massachusetts, about 40 miles from Cambridge. The challenges of turning this long-abandoned land into a real horse farm, including the restoration of the farmhouse and the barns as well as building a new house, are described in detail by Liz in the highly entertaining book *Seventy-Something Acres* (published by Cape Catley Ltd in 2006). Although Dave's skills in farming were not at the level of those in fluid mechanics, according to Liz, he enjoyed the day-to-day running of the farm (**Figure 2**).

1.5. Overview

The rest of this article focuses on David Benney's research in fluid mechanics. Benney made highly original, transformative contributions to a wide range of classical problems of nonlinear wave propagation and instability phenomena. In some instances, such as 3D shear flow instability first studied in his PhD thesis, his approach was way ahead of its time, and he later revisited the problem when his ideas and those of the fluids community had further matured. For this reason, rather than following a chronological order, we shall highlight the common threads of his work; namely, resonant nonlinear wave interactions, the use of multiple-scale asymptotic techniques to derive generic evolution equations for nonlinear wave dynamics, and the significance of 3D disturbances in nonlinear wave processes.

2. RESONANT NONLINEAR WAVE INTERACTIONS

In the 1950s, the state of the art in nonlinear water wave theory was limited to Stokes' (1847) solution for weakly nonlinear periodic waves on deep water and the classical works of Boussinesq (1871), Rayleigh (1876), and Korteweg & de Vries (1895) for nonlinear shallow-water waves. The

Figure 2

(*Left*) David and Elizabeth Benney's farmhouse built in the 1990s on their seventy-plus-acre farm in Upton, Massachusetts. (*Right*) Farmer Benney moves equipment.

next significant advance in the theoretical understanding of nonlinear effects in water waves, and nonlinear wave processes in general, was realized in the 1960s with the discovery of resonant nonlinear wave interactions. The idea has its origin in the attempts of Phillips (1960), Longuet-Higgins (1962), and Hasselmann (1962) to shed light on how nonlinear interactions among surface gravity wave modes on deep water may redistribute the initial energy in these modes. These early studies were based on straightforward perturbation expansions in wave steepness. After lengthy algebra, it became apparent that energy transfer is generally weak at second order, and the same is true at third order unless four modes, with wave vectors $\mathbf{k}_i$ and frequencies ω_i ($i = 1, \ldots, 4$) consistent with the dispersion relation $\omega^2 = g|\mathbf{k}|$, happen to satisfy the resonance conditions

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0, \quad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0. \tag{1}$$

Furthermore, the analysis suggested that three modes belonging to such a quartet may excite the fourth member resonantly; that is, the newly generated wave amplitude would grow linearly in time. As recounted by Phillips (1981), back then, the physical significance of these resonances was a matter of vigorous debate as, based on energy conservation, the linear amplitude growth cannot persist after long enough time; thus, it was not clear how much energy ultimately gets transferred to the newly generated wave.

The missing link was provided by Benney (1962), who recognized that resonantly interacting dispersive wave modes behave like weakly nonlinear oscillators with natural frequencies in internal resonance. For the latter case, it was already known that the resonance can cause significant energy transfer among the oscillators over a timescale of many periods, and the long-time dynamics of this energy-sharing process is described asymptotically by nonlinear amplitude equations that can be obtained by the newly developed technique of two-timing (Bogoliubov & Mitropolsky 1961). Adapting this approach to surface gravity wave interactions, Benney (1962) derived four coupled evolution equations for the complex amplitudes—an eighth-order system for the real amplitudes and phases—of four modes that satisfy the resonance conditions in Equation 1. In spite of the tedious algebra involved in the derivation (some of the analytic expressions in the paper are pages long) and the fact that the evolution equations could not be solved in closed form, the essential physics of a resonant interaction became clear by noting three simple integrals. These revealed that significant energy transfer does occur among modes forming a resonant quartet, in a way such that the growth of one member of the group is compensated by the decay of another. Furthermore, if only three members of a quartet are present initially, their interaction can generate the fourth member to the same level as the initial condition. However, this process takes a long time, namely $\mathcal{O}(1/\epsilon^2)$ wave periods, where $\epsilon \ll 1$ is the wave steepness.

The analysis of resonant quartets by Benney (1962) was pivotal to the ensuing rapid developments in the theory of nonlinear dispersive waves, and David Benney was a major contributor to this remarkable progress. Specifically, resonant triads, which are the lowest-order resonant interactions possible, can be treated in a similar fashion and amount to energy sharing on a faster, $\mathcal{O}(1/\epsilon)$–wave periods timescale. In water waves, resonant triads are not allowed by the dispersion relation when only gravity is present. However, resonant triads can arise in internal gravity waves and also play an important part in interactions between gravity and capillary wave modes on the surface of water. Furthermore, various instabilities of periodic waves to infinitesimal perturbations, such as the Benjamin–Feir modulational instability of surface waves and the parametric subharmonic instability of internal gravity waves, can be understood in terms of resonant interactions.

In addition, from a theoretical standpoint, Benney (1962) clarified how the linear-in-time (secular) growth predicted by straightforward (naive) perturbation expansions can be interpreted in terms of uniformly valid expansions involving a second (slow) timescale; this opened the way for the now prevalent use of multiple-scale techniques in nonlinear wave problems. Finally, the same theoretical approach using uniformly valid expansions with multiple timescales was applied by Benney and coworkers to weakly interacting random waves, where again resonant interactions are the dominant mechanism of energy transfer (Benney & Saffman 1966; Benney & Newell 1967a). These ideas laid the foundations for later studies of wave turbulence (e.g., Newell et al. 2001).

3. MODULATED WAVE TRAINS

3.1. The Nonlinear Schrödinger Equation

The prominent role of resonant wave groups (triads, quartets, etc.) in redistributing energy within a discrete spectrum of wave modes prompted Benney and his PhD student Alan Newell to inquire into how resonant nonlinear interactions may affect a continuous wave spectrum. Benney & Newell (1967b) addressed this question in the simplest case of a narrow-band spectrum, corresponding to wave packets that comprise a monochromatic carrier modulated by a slowly varying envelope in both space and time, and obtained equations for the propagation of the wave packet envelopes using a multiple-scale procedure. Among many novel aspects, this landmark paper demonstrated that such envelope evolution equations take a generic form that only depends on the dispersion relation, $\omega = \omega(k)$, and the coefficients of the nonlinear interaction terms in the amplitude equations for resonant triads, quartets, etc. Of particular significance is the equation governing the complex envelope $A(X, T)$ of a single wave packet,

$$i\mu(A_T + c_g A_X) = \mu^2 \beta A_{XX} + \epsilon^2 \delta A^2 A^*. \qquad 2.$$

Here, $(X, T) = \mu(x, t)$ are the scaled (slow) envelope variables, where $\mu \ll 1$ measures the spectral spread about the carrier, $c_g = \omega'(k)$ denotes the group velocity, and $\beta = -\omega''(k)/2$ is evaluated at the carrier wave number. The nonlinear coefficient δ derives from the self-interaction of the envelope, a degenerate form of a quartet interaction that causes an $\mathcal{O}(\epsilon^2)$ correction to the frequency of a uniform periodic wave of steepness $\epsilon \ll 1$, similar to the Stokes (1847) solution for surface gravity waves. The envelope Equation 2 accounts for the leading-order nonlinear and dispersive effects, and a balance is reached at $\epsilon = \mu$. Then, in a frame moving with $c_g, X' = X - c_g T$, both of these effects come into play on the same slow time, $T' = \epsilon^2 t$, and Equation 2 reduces to the NLS equation in standard form.

Benney & Newell (1967b) found exact solutions of the NLS equation in the form of periodic and solitary waves, as well as a similarity solution. Furthermore, they used the NLS equation to examine the stability of a uniform wave train (in which case, the envelope does not vary in X and is a special solution of the NLS equation) to sideband perturbations. Their analysis revealed instability for $\beta\delta > 0$ and stability for $\beta\delta < 0$. This criterion was consistent with that obtained independently by Benjamin (1967) based on discrete resonant quartet interactions, and by Whitham (1967) using a different approach. However, according to the NLS equation (and the discrete quartet stability analysis), the instability is confined to a finite range of sideband wave numbers, whereas Whitham's theory predicts an unbounded range of unstable sidebands. The reason for this discrepancy is that Whitham's stability analysis was based on modulation equations for slowly varying wave trains of finite amplitude; however, as noted by Chu & Mei (1970) and also discussed in section 15.5 of Whitham (1974), these finite-amplitude modulation equations do not recover the NLS equation in the small-amplitude limit owing to a missing dispersive term that, while formally negligible in the finite-amplitude theory, contributes at leading order in the weakly nonlinear–weakly dispersive regime. This underscores the fundamental significance of the NLS equation as the canonical envelope equation in the distinguished limit where dispersive and nonlinear effects are equally important.

3.2. Three-Dimensional Water Wave Packets and the Benney–Roskes System

The envelope of a gravity wave packet on water of finite depth h satisfies the NLS equation when modulations are restricted to the carrier propagation direction (e.g., Hasimoto & Ono 1972). For such 1D modulations, the criterion based on the sign of $\beta\delta$ mentioned above implies that Stokes periodic waves on relatively shallow water ($kh < 1.363$) are stable, while those on relatively deep water ($kh > 1.363$) are unstable, which is commonly termed the Benjamin–Feir instability (Benjamin & Feir 1967, Feir 1967). In the general case of oblique modulations, however, the NLS equation is replaced by a more complicated equation system that couples the propagation of the envelope to a long-wave component induced by the Reynolds stresses due to the presence of 2D envelope variations. Benney and his PhD student Gerald Roskes were the first to derive this equation system, which they also used to show that a uniform wave train is always unstable to oblique modulations except for the special case of $kh = 0.38$ (Benney & Roskes 1969). Apparently unaware of this earlier work, Davey & Stewartson (1974) derived a reduced version of the Benney–Roskes system, sometimes termed the Davey–Stewartson equations. The Davey–Stewartson equation system has received considerable attention by the mathematics community as a prototype model of nonlinear interactions between short and long waves.

3.3. Resonant Long–Short Wave Interactions

The coupled propagation of a 3D gravity wave packet with its induced mean flow, governed by the Benney–Roskes system, is an interaction of two disturbances with disparate length- and timescales. Other common examples of such long–short wave interactions are those between gravity waves and capillary ripples on the surface of water and between short-scale surface and long-scale internal gravity waves. Apart from their fundamental significance, long–short wave interactions play an important part in interpreting Synthetic Aperture Radar (SAR) images of the ocean surface—the return SAR signals are most sensitive to surface waves with wavelengths around 30 cm, and it is essential to know how such relatively short waves are modified by surface and internal gravity waves of longer scales (e.g., Beal et al. 1981).

Benney (1976, 1977) developed a simple, general condition for resonant energy transfer between long and short waves. He considered wave triads comprising two short waves, $\mathbf{k}_1 = \mathbf{k} + \Delta\mathbf{k}/2$ and $\mathbf{k}_2 = \mathbf{k} - \Delta\mathbf{k}/2$, and a long wave, $\mathbf{k}_3 = \Delta\mathbf{k}$, with $|\Delta\mathbf{k}| \ll |\mathbf{k}|$, which automatically satisfy $\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_3$. Then, from the frequency resonance condition, $\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3)$, to leading order in $\Delta\mathbf{k}$, it follows that short waves interact resonantly with long waves if the condition $\mathbf{c}_l \cdot \mathbf{c}_{gs} = |\mathbf{c}_l|^2$ is satisfied, that is, when the projection of the group velocity $\mathbf{c}_{gs}$ of the short wave in the direction of the long-wave phase velocity $\mathbf{c}_l$ is equal to $|\mathbf{c}_l|$. In the case of 1D wave propagation, this condition reduces to $c_{gs} = c_l$, which makes sense on physical grounds, given that the envelope of a short wave behaves as a long wave that moves with c_{gs} so that resonance would be expected if there are natural long-wave modes with the same speed. Apart from this intuitive physical appeal, the fact that the long–short wave resonance condition is a limiting form of a resonant triad plays an important part in the stability analysis of a short wave to large-scale modulations, as the modulations consistent with this resonance are preferentially amplified. This interesting nonlinear energy-transfer mechanism, first discussed by Benney (1976) for gravity–capillary surface waves, also finds applications in atmospheric internal gravity waves (Tabaei & Akylas 2007, Wilhelm et al. 2018).

4. NONLINEAR LONG WAVES

The first theoretical investigations of solitary waves go back to the nineteenth century. Following the now-famous observations by Scott Russell (1844) of a single hump of water propagating without change of shape along a Scottish canal, Boussinesq (1871) and Rayleigh (1876) found approximate expressions for solitary surface gravity waves. These were later confirmed by Korteweg & de Vries (1895) via the so-called KdV equation, an approximation to the full water wave problem that accepts traveling wave solutions in the form of solitary waves. Apart from the classical case of shallow-water waves, however, the wide applicability of the KdV equation was not fully recognized in the early 1960s, although KdV-type solitary waves had been known to arise in specific physical settings (e.g., Long 1956, Peters & Stoker 1960, Benjamin 1962).

Benney, by contrast, took a general approach to nonlinear long-wave motion, utilizing the presence of two disparate length scales in the problem, namely the fluid depth, h, and the typical wavelength, l. Thus, the long-wave parameter, $\mu = h/l \ll 1$, and the nonlinearity parameter, $\epsilon = a/h$, where a is the typical wave amplitude, are the main controlling parameters. Using perturbation techniques under various balances of μ and ϵ, he derived evolution equations for long-crested waves in a variety of flow configurations, ranging from inviscid waveguides to viscous film flow.

4.1. Unidirectional Propagation in an Inviscid Fluid Layer

Benney (1966a) studied weakly nonlinear long waves in an inviscid fluid layer of depth h in the presence of stratification, shear, and a free surface. This flow configuration is of fundamental interest in GFD and includes the classical shallow-water problem as a special case. The linear problem supports a countable infinity of wave modes, each of which satisfies its own dispersion relation determined from solving an eigenvalue problem. Furthermore, according to these dispersion relations, the long-wave limit ($\mu \to 0$) is nondispersive, so infinitesimal ($\epsilon \to 0$) disturbances of a certain mode propagate along the layer with the corresponding long-wave speed c regardless of the wavelength. Thus, assuming unidirectional propagation, to zeroth order in μ and ϵ, the mode amplitude profile $A(x,t)$ travels with c as a wave of permanent form in keeping with the 1D wave equation $A_t + cA_x = 0$, and it is natural to ask how weak nonlinear and dispersive effects influence the long-time dynamics of such a disturbance. To address this issue, Benney (1966a) expands the time-evolution of $A(x,t)$ in powers of μ and ϵ as

$$\begin{aligned} A_t = &-cA_x + \epsilon r(A^2)_x + \mu^2 sA_{xxx} + \epsilon^2 \lambda_1 (A^3)_x \\ &+ \epsilon\mu^2(\lambda_2 AA_{xxx} + \lambda_3 A_x A_{xx}) + \mu^4 \lambda_4 A_{xxxxx} + \ldots, \end{aligned} \qquad 3.$$

where r, s, and $\lambda_1, \ldots, \lambda_4$ are constants to be computed by suppressing secular terms at each order in the expansion. This novel perturbation procedure demonstrates that, under the balance $\epsilon = \mathcal{O}(\mu^2)$ where nonlinear and dispersive effects are equally important, the leading-order propagation of a weakly nonlinear long-wave mode is governed by the KdV equation. Thus, solitary waves with a sech^2 profile are to be expected in this general setting, and the expansion in Equation 3 makes it possible to compute higher-order effects beyond the KdV approximation (e.g., Skopovi & Akylas 2004). Furthermore, the same approach is applicable to weakly nonlinear–weakly dispersive wave propagation of a single mode, as well as interacting modes, in other flow configurations that act as waveguides (e.g., waves in a rotating fluid).

4.2. Thin-Film Flow Down an Inclined Plane

In a follow-up paper, Benney (1966b) applied a long-wave methodology to viscous flow of a thin film down an inclined plane. This study in fact arose out of consulting work for Polaroid, to which David Benney was a regular consultant for many years. Here, the basic state is the steady, parallel flow with uniform film thickness and parabolic velocity profile along the incline, driven by gravity. The onset of linear instability for this basic flow occurs at a critical Reynolds number (based on the undisturbed film thickness), $R = R_c = \mathcal{O}(1)$, for long-wave disturbances ($\mu \to 0$) that travel with constant speed regardless of wavelength (Benjamin 1957, Yih 1963). This suggests a long-wave weakly nonlinear ($\mu, \epsilon \ll 1$) stability analysis for $R \approx R_c$. Benney (1966b), however, takes a more general approach where the time-evolution of the film thickness $h(x,t)$ is expanded in powers of μ, keeping $\epsilon = \mathcal{O}(1)$ and $R = \mathcal{O}(1)$. To leading order, this long-wave expansion yields a nonlinear, kinematic wave equation, $h_t + c(h)h_x = 0$, which predicts wave breaking similar to shallow-water theory, and higher-order terms are computed by suppressing secular behavior in the expansion (although details are not given in the paper!). As these terms are generally nonlinear and involve higher-order spatial derivatives, they are expected to play a decisive role in wave breaking. In the weakly nonlinear regime, the finite-amplitude evolution equation for $h(x,t)$ reduces to a Burgers-type equation, which can be used to study nonlinear effects near the neutral stability curve. While rather brief (and somewhat cryptic), Benney's (1966b) was the first study of film flows that derived evolution equations (reduced model equations) from the full Navier–Stokes equations and free-surface conditions. This novel approach inspired many subsequent developments in modeling these fundamentally interesting and technologically important flows [e.g., see the review articles by Chang (1994) and Oron et al. (1997)].

4.3. Three-Dimensional Shallow-Water Waves and the Benney–Luke Equation

Following his analysis of resonant gravity wave quartets on deep water (Benney 1962), Benney with his Masters student Jon C. Luke (who later received his PhD at Caltech under Gerald B.

Whitham) set out to examine possible resonant interactions of gravity waves of permanent form, namely cnoidal and solitary waves, on shallow water. While such waves can be approximated by the KdV equation and higher-order extensions that follow from the expansion in Equation 3, these evolution equations are limited to unidirectional wave propagation and cannot be used to study interactions of waves propagating in arbitrary directions. To tackle this issue, using a systematic perturbation procedure in terms of the long-wave parameter μ and the nonlinearity parameter ϵ, Benney & Luke (1964) derived an evolution equation for 3D gravity waves on water of finite depth, commonly termed the Benney–Luke (BL) equation. The BL equation to leading order reduces to the classical wave equation for 3D linear, shallow-water waves but also accounts for weak nonlinear and dispersive effects in a consistent manner. Thus, for 1D waves, the BL equation recovers the KdV equation and also includes as a special case, the so-called KP equation (Kadomtsev & Petviashvili 1970), which assumes unidirectional propagation with weak transverse variations (e.g., Akylas 1994). Based on the BL equation, Benney & Luke (1964) found that interactions of waves of permanent form are generally weak unless these waves propagate in quasi-parallel directions, a conclusion that was confirmed in later detailed studies of oblique solitary wave interactions (Miles 1977a,b). Furthermore, the BL equation and its extensions that include surface tension and variable topography have been used to study various 3D wave phenomena in shallow water, including the reflection of obliquely incident solitary waves from a wall (Funakoshi 1980), solitary wave interactions on beaches (Ablowitz & Baldwin 2012), and forced generation of 3D (lump) solitary waves by flow over an obstacle (Berger & Milewski 2000).

5. SHEAR FLOW INSTABILITY

5.1. Three-Dimensional Effects Are Essential to Nonlinear Instability

As mentioned earlier, David Benney's PhD thesis, under the supervision of C.C. Lin, was in the area of hydrodynamic stability and was motivated by experiments on the instability and transition of the Blasius boundary layer, carried out at the US National Bureau of Standards (Klebanoff et al. 1962). These experiments emphasized the 3D nature of instability and transition. This underscored the limitations of linear stability theory based on the Orr–Sommerfeld equation, which naturally focuses on 2D disturbances (Tollmien–Schlichting waves), as the onset of linear instability is realized for 2D waves according to Squire's theorem. Specifically, Klebanoff et al. (1962) reported that an essentially 2D Tollmien–Schlichting wave, excited by a vibrating ribbon at a certain frequency, developed periodic spanwise (transverse) variations that became more pronounced as the wave traveled downstream and grew in amplitude. Furthermore, the spanwise variations in the primary wave were accompanied by a secondary, spanwise-periodic mean flow in the form of longitudinal (streamwise) vortices. To explain these observations, Benney & Lin (1960) and Benney (1961) proposed a simple theoretical model using for convenience a hyperbolic-tangent basic flow profile, and Benney (1964) extended this to a broken-line boundary layer profile. The model considers a three-wave system that comprises a 2D nearly neutral wave and two oblique waves with the same frequency and streamwise wave number as the 2D wave, but with opposite spanwise wave numbers, forming a standing wave in the transverse direction. The main result was that the induced mean flow due to quadratic nonlinear interactions of these three wave components features spanwise-periodic streamwise rolls akin to those observed experimentally by Klebanoff et al. (1962). In spite of this agreement, the Benney–Lin theory also received criticism (e.g., Stuart 1962), since their assumption that 2D and 3D wave components have the same streamwise wave number and frequency is not justified for the Blasius boundary layer, and accounting for the presence of a frequency mismatch introduces a temporal modulation effect (Antar & Collins 1975) that is not supported by the experiments. However, according to Maslowe (1981, p. 220), "This theory reproduces so many features of the experiments that it seems highly likely that the basic idea is correct."

Maslowe's conjecture has been confirmed thanks to various contributions by Benney and others in the last four decades. First and foremost, Benney (1984) derived a closed nonlinear equation system that governs the coupled evolution of an $\mathcal{O}(1)$ 3D mean shear flow, whose dominant component is in the streamwise (x-) direction but where all three components vary in the transverse (z-) direction, and an $\mathcal{O}(\epsilon)$ 3D disturbance with harmonic [$\propto \exp i\alpha(x - ct)$] streamwise dependence. This strong mean flow–first harmonic (MFFH) coupling arises because (in the inviscid limit) the mean flow is a degenerate zero-frequency mode and forms essentially a resonant triad with the first harmonic. Benney (1984) assumed inviscid flow conditions, but this MFFH coupling mechanism applies generally at high Reynolds numbers, $R = \mathcal{O}(1/\epsilon)$, as shown later by Benney & Chow (1989). Thus, the MFFH evolution equations afford a generic, uniformly valid description of the 3D nonlinear shear flow instability processes suggested by the Benney–Lin theory, without the severe limitations of the earlier work. More importantly, however, the MFFH theory establishes that $\mathcal{O}(\epsilon)$ 3D disturbances can have an $\mathcal{O}(1)$ effect on the underlying shear flow, which from 2D quickly becomes 3D.

Early studies based on the MFFH evolution equations (Benney 1984; Benney & Chow 1985, 1989) focused on the linearized stability to 3D perturbations of a 2D state comprising the basic shear flow and a 2D neutral mode. These analyses parallel Navier–Stokes numerical investigations of 3D instabilities of 2D finite-amplitude periodic states in wall-bounded shear flows (e.g., Orszag & Patera 1983). However, as noted by Benney (1984, p. 14), the MFFH equations "are of far more fundamental importance than any linear stability problem they may contain."

The precise role of the MFFH interaction mechanism in shear flow instability and transition was elucidated in more recent work by Fabian Waleffe and coworkers. Motivated by Benney (1984) and numerical simulations of near-wall turbulence structures (Hamilton et al. 1995), Waleffe (1995a,b; 1997) proposed a self-sustaining 3D nonlinear instability process in shear flows, which has also been tied to the existence of exact 3D coherent structures in the form of steady and traveling wave states in plane Couette and Poiseuille flow (e.g., Waleffe 1998). At high Reynolds numbers, this process involves (*a*) $\mathcal{O}(1)$ streaks, that is, spanwise variations of the mean streamwise velocity; (*b*) $\mathcal{O}(1/R)$ mean flow components in the vertical and transverse directions, forming streamwise rolls; and (*c*) an $\mathcal{O}(1/R)$ 3D mode with harmonic variation in the streamwise direction. These three elements are also the key ingredients of the MFFH interaction mechanism. Furthermore, numerical computations based on the Navier–Stokes equations for plane Couette flow at high Reynolds number (Waleffe & Wang 2005) find a lower branch of exact coherent structures that comprise streaks, streamwise rolls, and a 3D eigenmode, consistent with the MFFH theory; these coherent states are inherently nonlinear (do not bifurcate from the laminar flow) and provide an $\mathcal{O}(1/R)$ threshold for transition.

5.2. Nonlinear Critical Layer

Another far-reaching contribution of David Benney to shear flow instability is the theory of the nonlinear critical layer. In the linear stability analysis of parallel shear flows, critical layers arise where the phase speed of a neutral sinusoidal disturbance happens to match the basic flow speed. These are special points of the Rayleigh equation (the inviscid limit of the Orr–Sommerfeld equation), as the solution features a logarithmic branch point there, and some additional physics is needed to heal this singular behavior. Furthermore, how the singularity is resolved determines whether the Reynolds stress associated with the disturbance suffers a jump at the critical level, which in turn bears on the flow stability characteristics. The traditional way to proceed is by viewing neutral modes of the Rayleigh equation as limits of Orr–Sommerfeld modes at high Reynolds numbers, $\alpha R \to \infty$, where α is the disturbance wave number (e.g., Drazin & Reid 1982). In this linear, viscous approach, the singularity is thus handled by inserting around the critical level a layer of thickness $\mathcal{O}(\gamma)$ [$\gamma = (\alpha R)^{-1/3} \ll 1$], where viscous effects are important, and it turns out that the logarithm in the Rayleigh solution experiences a phase change of $-\pi$ from above to below the critical level.

The possibility of an alternative approach, where nonlinearity rather than viscosity is employed to resolve the flow near a critical level, was first mentioned by Lin & Benney (1964), and a detailed study was presented five years later by Benney and his PhD student Robert Bergeron (Benney & Bergeron 1969). A nonlinear critical layer has $\mathcal{O}(\epsilon^{1/2})$ thickness, where $\epsilon \ll 1$ is the disturbance amplitude parameter, and to leading order features the characteristic cat's eye flow pattern also present in the viscous solution. However, higher harmonics are no longer ordered, implying a strong nonlinear balance, and more importantly, the phase change of the solution across the critical level vanishes. This leads to a different eigenvalue problem and a new class of neutral modes from the classical viscous theory. Later, Richard Haberman, another PhD student of Benney, studied the more general situation where nonlinear and viscous effects are equally important inside the critical layer ($\gamma \sim \epsilon^{1/2}$), and he showed that the phase change varies monotonically from $-\pi$ in the viscous regime ($\gamma \gg \epsilon^{1/2}$) to zero in the nonlinear regime ($\epsilon^{1/2} \gg \gamma$) (Haberman 1972). Furthermore, Benney (1983) pointed out that wave packet effects, too, can be a significant factor in a critical layer, and Maslowe et al. (1994) explored the situation where these effects dominate over nonlinearity and viscosity, in which case the phase change turns out to be $-\pi$.

Apart from their theoretical significance, nonlinear critical layers are also of physical interest, particularly in GFD applications where Reynolds numbers are typically high due to the stabilizing effects of stratification and rotation [e.g., see the reviews by Maslowe (1986, 2014)]. While critical layer theories assume steady or nearly steady flow conditions, more recent numerical studies show how neutral modes with nonlinear critical layers can be generated in a geophysical setting (e.g., Maslowe & Clarke 2002). Nonlinear critical layers are also important in the propagation of solitary Rossby waves in a horizontally sheared zonal flow (e.g., Redekopp 1977, Caillol & Grimshaw 2007).

6. CONCLUDING REMARKS

David Benney was a truly original thinker who made transformative contributions to our understanding of nonlinear wave and instability processes in fluid flows. Most notably, (*a*) he introduced a new, powerful paradigm of theoretical modeling based on asymptotic evolution equations (reduced models) that capture the essential physics and are widely applicable under various flow conditions. (*b*) His work is characterized by remarkable breadth. Particularly in the field of nonlinear wave motion, he was at the forefront in all major advances since the 1960s (resonant wave interactions, modulated wave trains, long-crested waves, 3D effects, etc.), except for the development of inverse scattering, an ingenious technique for solving certain nonlinear evolution equations; however, two of his PhD students (Alan Newell and Mark Ablowitz) have made seminal contributions to this more mathematical aspect. (*c*) His work has also had a lasting impact in various fields other than fluid mechanics, including meteorology, oceanography, optics, and plasma physics.

Perhaps more importantly, however, Benney was a creative scientist of the utmost integrity. He led by example, adhering to the highest scientific standards and without seeking personal recognition. As Alan Newell noted in the obituary for Benney published by the MIT mathematics department (MIT Math. Dep. 2015), "Dave was a modest man who had little to be modest about.

With his gentle, self-effacing manner and humor, he tended to deflect any superlatives and accolades aimed in his direction. But in truth, he was a first rate leader: generous to all, regardless of rank, he had a strong moral compass, a principled view of life and a backbone of steel when it came to doing the right thing."

DISCLOSURE STATEMENT

The author is not aware of any biases that might be perceived as affecting the objectivity of this review.

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