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Computational Methods in Environmental and Resource Economics

Yongyang Cai

Department of Agricultural, Environmental and Development Economics, The Ohio State University, Columbus, Ohio 43210, USA; email: cai.619@osu.edu

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Abstract

Computational methods are required to solve problems without closed-form solutions in environmental and resource economics. Efficiency, stability, and accuracy are key elements for computational methods. This review discusses state-of-the-art computational methods applied in environmental and resource economics, including optimal control methods for deterministic models, advances in value function iteration and time iteration for general dynamic stochastic problems, nonlinear certainty equivalent approximation, robust decision making, real option analysis, bilevel optimization, solution methods for continuous time problems, and so on. This review also clarifies the so-called curse of dimensionality, and discusses some computational techniques such as approximation methods without the curse of dimensionality and time-dependent approximation domains. Many existing economic models use simplifying and/or unrealistic assumptions with an excuse of computational feasibility, but these assumptions might be able to be relaxed if we choose an efficient computational method discussed in this review.

1. INTRODUCTION

In environmental and resource economics, problems often have no closed-form solutions, so we have to rely on computational methods to solve them and then provide economic analysis. While analytic results can be helpful in building intuition, they have to make many strong and simplifying assumptions, so sometimes their closed-form solutions may result in economic analysis that is more confusing than clarifying.

This review focuses on computational methods, particularly general and recent methods, more than substantive modeling and applications, although I provide some references to applied research that focuses on environmental and resource economics. The review also focuses on solution methods for discrete time dynamic stochastic problems, particularly new advances and clarifications in the most popular method—value/policy function iteration. I also review other methods, including NLCEQ (nonlinear certainty equivalent approximation) (Cai et al. 2017c), robust decision making, and bilevel optimization. For general background, traditional methods, and standard rules in computational methods for economists, see Rust (1996), Judd (1998), Ljungqvist & Sargent (2000), Miranda & Fackler (2002), Bertsekas (2005, 2007), and Cai & Judd (2014) for details. This review will also not discuss agent-based models [see Farmer et al. (2015) for a detailed discussion about agent-based models in climate change economics] or econometric and statistical methods.[1]

The review is organized as follows. Section 2 discusses optimal control methods for deterministic models, including solving a social planner's problem and finding equilibrium under no uncertainty. Section 3 presents value function iteration, the most popular method for solving discrete time dynamic stochastic programming problems under a social planner's preference. Section 4 presents time iteration, another popular method for finding dynamic general equilibrium of discrete time dynamic stochastic programming problems. While both value function iteration and time iteration are for general dynamic programming problems that may have a finite or infinite time horizon and may be nonstationary, many economic problems study infinite-time-horizon stationary problems, which Section 5 specifically discusses. Section 6 briefly reviews computational methods for robust decision-making problems. Section 7 discusses other computational methods including NLCEQ, approximate dynamic programming, real options pricing, and bilevel optimization for solving principal-agent problems. Section 8 briefly reviews computational methods for continuous time dynamic programming problems. Section 9 provides detailed discussions of the curse of dimensionality, boundedness, Monte Carlo techniques, approximation, and stopping criteria. Section 10 concludes. The **Supplemental Appendix** discusses the complete Chebyshev approximation, simplicial complete Chebyshev approximation, and the Chebyshev regression algorithm; it also provides an illustrative example of value function iteration with complete Chebyshev approximation and its code for solving a simple optimal growth problem.

Supplemental Material >

2. OPTIMAL CONTROL METHODS FOR DETERMINISTIC MODELS

2.1. Social Planner's Problem

Most deterministic discrete time dynamic programming (DP) problems in environmental and resource economics can be written as

[1] In econometrics, there are many computational issues that may lead to different, even opposite, solutions. For example, Cafiero et al. (2011, 2015) show that using a much finer grid to approximate the equilibrium price function leads to positive evidence for the role of storage arbitrage, contrary to a previous claim of Deaton & Laroque (1995, 1996) using a coarse grid. Guerra et al. (2015) show serious differences in magnitudes of practical interest between using annual price data and using December price data for testing a storage model.

$$\max \quad \sum_{t=0}^{T-1} \beta^t u_t(\mathbf{x}_t, \mathbf{a}_t) + \beta^T V_T(\mathbf{x}_T) \qquad 1.$$

$$\text{s.t.} \quad \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t),\ t = 0, 1, \ldots, T-1$$

$$\mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t)$$

$$\mathbf{x}_0 \text{ given},$$

where t is time (period), T is the time horizon (which can be infinite), $\mathbf{x}_t$ is a vector of state variables (e.g., capital or resource stock), $\mathbf{a}_t$ is a vector of decision variables (e.g., consumption), $\mathbf{f}_t$ is a vector of functions representing transition laws of the state variables, $\beta < 1$ is the discount factor, u_t is the social planner's utility function, V_T is the terminal value function (for infinite-horizon problems, V_T is zero everywhere), and $\mathcal{D}_t(\mathbf{x}_t)$ is the feasible set for decision variables at time t.

The optimal control method is the most common method to solve the deterministic problem (Equation 1) with a finite horizon. That is, if the functions and variables are continuous, then we view Equation 1 as a nonlinear programming (NLP) problem with variables $(\mathbf{x}_1, \ldots, \mathbf{x}_T, \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{T-1})$ and constraints $\mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{a}_t)$ and $\mathbf{a}_t \in \mathcal{D}_t(\mathbf{x}_t)$ for $t = 0, 1, \ldots, T-1$, and then use an NLP solver to solve it directly;[2] if the functions and variables are discrete, then we can use integer programming to solve Equation 1; and if some but not all of the functions and variables are discrete, then we can use mixed integer nonlinearly constrained optimization (MINLP) solvers to solve Equation 1. In MATLAB, we can use the fmincon solver (a solver for finding a minimum of a constrained optimization problem) to solve Equation 1 after transforming it into a minimization problem by multiplying the objective by -1, and in cases where fmincon does not work well, we use an alternative solver such as KNITRO. However, GAMS (General Algebraic Modeling System) or AMPL (A Mathematical Programming Language) can provide a more flexible environment than MATLAB, as there are more professional solvers available using GAMS or AMPL (see McCarl et al. 2018 or Fourer et al. 2003 for their user guides). For example, CONOPT (Drud 1994) is often more reliable and efficient than fmincon in solving NLP problems, and SNOPT (Gill et al. 2005) is another good alternative. The NEOS server (Czyzyk et al. 1998) (**https://neos-server.org/neos/solvers/index.html**) provides a long list of free solvers for various optimization problems (e.g., NLP, MINLP, global optimization) that can run GAMS or AMPL code. The flexibility of GAMS or AMPL is more useful for dealing with challenging optimization problems such as global optimization or problems with a flat objective over some decision variables, as we can try different solvers with the same code written in GAMS or AMPL to solve Equation 1 or even verify the accuracy of a solution of Equation 1 obtained from another solver. A good initial guess, scaling, and stopping criteria are also important for solving challenging problems.

For deterministic problems (Equation 1) with an infinite horizon, usually one can also use the optimal control method after truncating the infinite series at a large finite period T, because $\beta^t \to 0$ as $t \to \infty$ and infinite-horizon problems often require a transversality condition. In fact, if the utility function is a power function with a marginal elasticity larger than 1, then it is upper-bounded at zero, so $\beta^t u_t(\mathbf{x}_t, \mathbf{a}_t) \to 0$ for a typical economic problem.

2.2. Find Equilibrium

While DP problems usually focus on intertemporal equilibrium and long-term effects (often with hundreds of periods), another direction is to study short-term effects and spatial or sectoral

[2]Linear programming (LP) is a special case of NLP and usually is easier to solve with LP solvers such as CPLEX and Gurobi, so we do not discuss LP in this article.

equilibrium, which typically includes many regions or sectors (up to hundreds of sectors). The Global Trade Analysis Project (GTAP) founded by Thomas Hertel at Purdue University is one representative example (see Hertel 2013 for an overview of GTAP), and many other computable general equilibrium (CGE) models are presented by Dixon & Jorgenson (2013).[3] GTAP applies on the GEMPACK (General Equilibrium Modeling Package) platform, which is compared with GAMS and MPSGE [Mathematical Programming System for General Equilibrium (Rutherford 1999)] by Horridge et al. (2013). GTAP is also combined with other models in the literature. For example, Golub et al. (2009) extend GTAP to GTAP-AEZ-GHG, a general equilibrium framework, to model forest carbon sequestration and land management in agriculture and forestry, and Golub et al. (2013) extend GTAP-AEZ-GHG to study climate policy impacts.

For multi-regional, multi-sectoral, and/or multi-agent static problems, each region, sector, or agent is assumed to optimize their objectives with budget and market-clearing constraints and trade between regions, sectors, and/or agents. The decentralized equilibrium can be solved using a system of first-order conditions (and constraints). That is, the problem is to find a solution to the system of equations and inequalities

$$\begin{cases} \mathbf{F}(\mathbf{a}) = 0 \\ \mathbf{G}(\mathbf{a}) \geq 0, \end{cases} \tag{2.}$$

where $\mathbf{a}$ contains prices, quantities of (intermediate) products, and resource allocations. This may be solved in MATLAB using fsolve or other equation solvers. The system (Equation 2) may also be solved as a degenerate optimization problem,

$$\begin{aligned} \max_{\mathbf{x}} \quad & 1 \\ \text{s.t.} \quad & \mathbf{F}(\mathbf{a}) = 0 \\ & \mathbf{G}(\mathbf{a}) \geq 0, \end{aligned} \tag{3.}$$

using an optimization solver such as CONOPT and SNOPT to find a feasible point using the optimal control method. Sometimes the system (Equation 2) may contain complementarity conditions (such as $x_i x_j = 0$ with $x_i, x_j \geq 0$), which often make it challenging to solve. The mixed complementarity problems may be solved using MILES (Rutherford 1993, 1995) or PATH (Dirkse & Ferris 1995, Ferris & Munson 2000).

For deterministic dynamic problems, we can use the same computational methods to solve the following system of equations and inequalities:

$$\begin{cases} \mathbf{F}_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{a}_t, \mathbf{a}_{t+1}) = 0, \ \forall t \\ \mathbf{G}_t(\mathbf{x}_t, \mathbf{a}_t) \geq 0, \ \forall t, \end{cases} \tag{4.}$$

where $\mathbf{x}_t$ are state variables; $\mathbf{a}_t$ are other variables, including decision variables and prices; $\mathbf{F}_t$ represents transition laws of states, Euler equations, first-order conditions, and other equality constraints; and $\mathbf{G}_t$ represents all inequality constraints. For example, Baldwin et al. (2018) use the degenerate optimization method (Equation 3) but with the constraints in Equation 4 to obtain decentralized equilibrium when there is no carbon tax in the dirty energy sector or a subsidy in the renewable energy sector. One disadvantage of this method is that we have to derive the Euler equations and first-order conditions, and the system (Equation 4) may be challenging to solve.

[3] Also see Bergman (2005) for CGE modeling in environmental policy and resource management.

3. VALUE FUNCTION ITERATION

Most stochastic discrete time dynamic programming problems in environmental and resource economics can be written as

$$\begin{aligned} \max \quad & \mathbb{E}\left\{\sum_{t=0}^{T-1}\beta^t u_t(\mathbf{x}_t,\mathbf{a}_t)+\beta^T V_T(\mathbf{x}_T)\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1}=\mathbf{f}_t(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t),\ t=0,1,\ldots,T-1 \\ & \mathbf{a}_t\in\mathcal{D}_t(\mathbf{x}_t) \\ & \mathbf{x}_0 \text{ given,} \end{aligned} \tag{5.}$$

where $\mathbb{E}$ is the expectation operator and $\boldsymbol{\epsilon}_t$ is a vector of random variables at time t. Note that some state variables may have deterministic transition laws, e.g., the transition law of the j-th state variable is $x_{t+1,j}=g_{t,j}(\mathbf{x}_t,\mathbf{a}_t)$, but here we simplify our notation by defining $f_{t,j}(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t)=g_{t,j}(\mathbf{x}_t,\mathbf{a}_t)+0\cdot\boldsymbol{\epsilon}_t$ so that $x_{t+1,j}=f_{t,j}(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t)$. For simplicity in discussion of computational methods, we assume $\mathbf{x}_t$ are continuous variables (as discrete state variables can be simply added).

Value function iteration (VFI) is the most common method to solve Equation 5. It transforms Equation 5 to the following Bellman equation (Bellman 1957):

$$\begin{aligned} V_t(\mathbf{x}_t)=\max_{\mathbf{a}_t\in\mathcal{D}_t(\mathbf{x}_t)} \quad & u_t(\mathbf{x}_t,\mathbf{a}_t)+\beta\mathbb{E}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1}=\mathbf{f}_t(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t) \end{aligned} \tag{6.}$$

for $t=0,1,\ldots,T-1$, where $\mathbb{E}_t$ is the expectation operator conditional on the time-t information. With a given terminal value function V_T for finite-horizon problems, it iterates backward over time to get all value functions and policy functions. For infinite-horizon problems, we can truncate it at a finite horizon T and choose

$$V_T(\mathbf{x}_T)\approx\mathbb{E}_T\left\{\sum_{t=T}^{\infty}\beta^{t-T}u_t(\widetilde{\mathbf{x}}_t,\widetilde{\mathbf{a}}_t)\right\},$$

where $\widetilde{\mathbf{a}}_t$ and $\widetilde{\mathbf{x}}_t$ are a series of guessed decisions and states starting from the terminal state $\mathbf{x}_T$. A criterion in setting a terminal value function V_T is that a reasonable change in its terminal values (e.g., 10% up or down) will not result in a nonnegligible change in the solutions at the periods of interest.

Each iteration of Equation 6 contains three main parts: approximation, optimization, and integration. These parts are computed numerically for problems with continuous state variables, continuous random variables, and continuous decision variables. That is, with a given next-period value function approximation $\widehat{V}_{t+1}(\mathbf{x}_{t+1};\mathbf{b}_{t+1})$, numerical VFI constructs the current-period value function $\widehat{V}_t(\mathbf{x}_t;\mathbf{b}_t)$ by solving

$$\begin{aligned} \widehat{V}_t(\mathbf{x}_t;\mathbf{b}_t)\approx\widehat{\max_{\mathbf{a}_t\in\mathcal{D}_t(\mathbf{x}_t)}} \quad & u_t(\mathbf{x}_t,\mathbf{a}_t)+\beta\widehat{\mathbb{E}}_t\left\{\widehat{V}_{t+1}(\mathbf{x}_{t+1};\mathbf{b}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1}=\mathbf{f}_t(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t), \end{aligned} \tag{7.}$$

where hatted variables refer to the numerical implementation of approximation, optimization, and integration, and $\mathbf{b}$ is a vector of approximation coefficients. A typical numerical approximation of

a value function V is

$$\widehat{V}(\mathbf{x};\mathbf{b}) = \sum_j b_j \phi_j(\mathbf{x}), \tag{8.}$$

where $\{\phi_j(\mathbf{x})\}$ are basis functions (e.g., Chebyshev basis polynomials discussed in Section 9.4, or ordinary basis polynomials: $1, x, x^2, x^3, \ldots$, for univariate problems) and $\mathbf{b} = \{b_j\}$ are approximation coefficients [see Judd (1998) and Miranda & Fackler (2002) for a detailed discussion]. Moreover, a numerical approximation is often associated with a bounded approximation domain in the state space, which could be time variant. To solve Equation 7, we choose approximation nodes $\{\mathbf{x}_{t,i}\}$, solve the maximization problem

$$\begin{aligned} v_{t,i} = \widehat{\max_{\mathbf{a}_{t,i}\in\mathcal{D}_t(\mathbf{x}_{t,i})}} \quad & u_t(\mathbf{x}_{t,i},\mathbf{a}_{t,i}) + \beta\widehat{\mathbb{E}}_t\left\{\widehat{V}_{t+1}(\mathbf{x}_{t+1,i};\mathbf{b}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1,i} = \mathbf{f}_t(\mathbf{x}_{t,i},\mathbf{a}_{t,i},\boldsymbol{\epsilon}_t) \end{aligned} \tag{9.}$$

for every $\mathbf{x}_{t,i}$, and then find $\mathbf{b}_t$ such that $\widehat{V}_t(\mathbf{x}_{t,i};\mathbf{b}_t) \approx v_{t,i}$ for all i. Numerical optimization solvers include CONOPT, SNOPT, NPSOL (Gill et al. 1994), KNITRO (Byrd et al. 2006), or fmincon in MATLAB. Numerical integration methods include Gaussian quadrature rules (e.g., Gauss-Hermite quadrature for normal or log-normal distributions and Gauss-Legendre quadrature for uniform distributions) and monomial quadrature rules. See Judd (1998) for a detailed discussion of these.

The Bellman equation (Equation 6) applies widely to problems with separate utility functions. There are numerous applications in the literature; here I provide only several recent examples in environmental and resource economics. Daigneault et al. (2010) use the Bellman equation to find optimal forest management with fire risk and carbon sequestration credits. Cai et al. (2015a) and Lontzek et al. (2015) apply VFI to solve for the optimal carbon tax with an integrated assessment model with climate tipping risks.

Recently, recursive utility (Epstein & Zin 1989) has been used in DP. For example, Jensen & Traeger (2014) and Cai et al. (2016b, 2017b, 2018a) employ recursive utility in climate change economics for solving dynamic stochastic integrated assessment models (IAMs). They still apply VFI but with the more general Bellman equation

$$\begin{aligned} V_t(\mathbf{x}_t) = \max_{\mathbf{a}_t\in\mathcal{D}_t(\mathbf{x}_t)} \quad & u_t(\mathbf{x}_t,\mathbf{a}_t) + \beta\mathcal{G}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t,\mathbf{a}_t,\boldsymbol{\epsilon}_t), \end{aligned} \tag{10.}$$

where

$$\mathcal{G}_t\left\{V_{t+1}(\mathbf{x}_{t+1})\right\} \equiv \frac{1}{1-1/\psi}\left(\mathbb{E}_t\left\{((1-1/\psi)V_{t+1}(\mathbf{x}_{t+1}))^{\frac{1-\gamma}{1-1/\psi}}\right\}\right)^{\frac{1-1/\psi}{1-\gamma}}$$

with ψ and γ as the intertemporal elasticity of substitution and the risk aversion coefficient, respectively. Without loss of generality, we will use the Bellman equation (Equation 6) for later discussion.

VFI is also used to solve problems with learning. For example, Kelly & Kolstad (1999) employ VFI with neural network approximation to solve climate change economics problems with Bayesian learning. Leach (2007) extends their pioneering work to the case of learning about two correlated uncertain parameters. Recently, Kelly & Tan (2015) investigated the impact of learning an important uncertain parameter, climate sensitivity (which measures the temperature increase in

equilibrium if carbon concentration in the atmosphere doubles), on optimal climate policy under fat-tailed uncertainty about climate change. Rudik (2016) combines Bayesian learning and robust control in the context of optimal carbon taxation.

It is always important to check whether a computational method and code are actually solving problems with the desired accuracy. Cai et al. (2017b, 2018b) present how to verify and measure errors appropriately for numerical solutions from VFI. One technique for verification is to use the same code and the same computational methods (including the same approximation domains and nodes) for a stochastic problem to replicate a solution of its corresponding deterministic model, obtained by another programming language and/or another computational method (e.g., GAMS and the optimal control method). Another way is to check if higher-order approximations with wider approximation domains or higher-order quadrature rules will change results significantly. Normalized Euler equation errors (see, e.g., Cai et al. 2017c) and approximation errors are also important for measuring errors (Cai et al. 2018b).

4. TIME ITERATION

Time iteration, another backward iteration method, is also popular. It constructs policy functions with state variables as arguments, while their approximation domains are the same across decision variables. With next-period policy functions, we compute current-period policy functions by solving a system of intertemporal Euler equations and transition laws, and temporal first-order conditions and constraints. That is, at time t, with given next-period policy functions $\mathbf{A}_{t+1}(\mathbf{x}_{t+1})$, instead of solving the maximization problem (Equation 6) in VFI we solve the following system of equations and inequalities:

$$\begin{cases} \mathbf{E}_t(\mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{A}_t(\mathbf{x}_t), \mathbf{A}_{t+1}(\mathbf{x}_{t+1})) = 0 \\ \mathbf{F}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) = 0 \\ \mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t), \boldsymbol{\epsilon}_t) \\ \mathbf{G}_t(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0, \end{cases} \quad 11.$$

where $\mathbf{x}_t$ are state variables, $\mathbf{A}_t(\mathbf{x}_t)$ are policy functions, $\mathbf{E}_t$ represents Euler equations, $\mathbf{F}_t$ represents first-order conditions, $\mathbf{f}_t$ represents transition laws of state variables, and $\mathbf{G}_t$ represents all other constraints (e.g., nonnegativity constraints and complementarity constraints).[4] To solve Equation 11 numerically, we start with a given numerical approximation of next-period policy functions, $\widehat{\mathbf{A}}_{t+1}(\mathbf{x}_{t+1}; \mathbf{B}_{t+1})$, where $\mathbf{B}_{t+1}$ are approximation coefficients for decision variables; choose approximation nodes $\{\mathbf{x}_{t,i}\}$; solve

$$\begin{cases} \mathbf{E}_t(\mathbf{x}_{t,i}, \mathbf{x}_{t+1,i}, \mathbf{a}_{t,i}, \widehat{\mathbf{A}}_{t+1}(\mathbf{x}_{t+1,i}; \mathbf{B}_{t+1})) = 0 \\ \mathbf{F}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}) = 0 \\ \mathbf{x}_{t+1,i} = \mathbf{f}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}, \boldsymbol{\epsilon}_t) \\ \mathbf{G}_t(\mathbf{x}_{t,i}, \mathbf{a}_{t,i}) \geq 0 \end{cases} \quad 12.$$

for every $\mathbf{x}_{t,i}$; and then find $\mathbf{B}_t$ such that $\widehat{\mathbf{A}}_t(\mathbf{x}_{t,i}; \mathbf{B}_t) \approx \mathbf{a}_{t,i}$ for all i.

For stationary infinite-horizon problems, time iteration is often faster than VFI. However, most economic problems have monotone and concave value functions. For such problems, solving the system in Equation 12 may be more challenging than solving the maximization problem in

[4] An equality constraint $g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) = 0$ can be represented as a combination of two inequality constraints: $g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0$ and $-g(\mathbf{x}_t, \mathbf{A}_t(\mathbf{x}_t)) \geq 0$.

Equation 6 using VFI, as the first-order conditions and Euler equations may lose the convexity of the maximization problem in Equation 6.

Time iteration has been widely used in the literature. For example, Judd et al. (2014) incorporate the Smolyak method (Smolyak 1963) into time iteration, and Brumm & Scheidegger (2017) introduce an adaptive sparse grid method into time iteration, so that time iteration can solve large-dimensional problems.

In particular, time iteration is a typical method for nonstationary DSGE (dynamic stochastic general equilibrium) or dynamic stochastic game problems with multiple regions, sectors, and/or agents. That is, at each time, time iteration solves the system in Equation 11 to get the optimal resource allocation and investment among regions, sectors, and/or agents, as well as prices and quantities of goods, and then iterates backward until the initial time (for finite-horizon problems) or until convergence (for infinite-horizon stationary problems).

5. METHODS FOR STATIONARY INFINITE-HORIZON PROBLEMS

For stationary infinite-horizon dynamic problems (i.e., all functions and exogenous parameters are independent of time), the Bellman equation (Equation 6) becomes

$$V(\mathbf{x}) = \max_{\mathbf{a}\in\mathcal{D}(\mathbf{x})} \; u(\mathbf{x},\mathbf{a}) + \beta\mathbb{E}\left\{V(\mathbf{x}_+)\right\} \qquad 13.$$
$$\text{s.t.} \quad \mathbf{x}_+ = \mathbf{f}(\mathbf{x},\mathbf{a},\epsilon),$$

where $\mathbf{x}_+$ is the next-period state transited from current state $\mathbf{x}$, and we can choose an initial guess for the value function V and then iterate until VFI converges.

Let $\widehat{V}_k(\mathbf{x};\mathbf{b}_k)$ be the value function approximation at the k-th iteration. We assume VFI converges numerically if two consecutive value function approximations are sufficiently close, i.e.,

$$\left\|\widehat{V}_k - \widehat{V}_{k+1}\right\| < \varepsilon$$

for some functional norm $\|\cdot\|$ and a small positive number ε. The $\mathcal{L}^\infty$ norm is often used, i.e.,

$$\max_{\mathbf{x}} \left|\widehat{V}_k(\mathbf{x};\mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x};\mathbf{b}_{k+1})\right| < \varepsilon.$$

Numerically, we can replace the above formula with

$$\max_{i} \left|\widehat{V}_k(\mathbf{x}_i;\mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x}_i;\mathbf{b}_{k+1})\right| < \varepsilon$$

over a large-size set of points $\{\mathbf{x}_i\}$ on the state space. For example, Cai et al. (2017b, 2018b) use 1,000 Monte Carlo points in the state space of continuous state variables for every discrete state. We should also pay attention to the magnitude of $\widehat{V}$, as too large a magnitude will make it too challenging or time consuming to stop, and too small a magnitude will make VFI stop too early, with large errors. Thus, typically we first scale utility by a constant such that the value function has a reasonable magnitude. Alternatively we can use the difference of two consecutive policy functions as a substitute for the difference of two consecutive value functions. In addition, the value of the discount factor also matters, because a discount factor close to 1 implies a small time increment. Since one period's utility could then have little contribution to the objective function, which may make VFI stop too early, we often use $\varepsilon/(1-\beta)$ instead of ε.

However, for infinite-horizon stationary DSGE problems, perturbation methods (see, e.g., Judd & Guu 1993) and projection methods (see, e.g., Judd 1992) may be more efficient, although

perturbation methods can only provide locally accurate solutions around the nonstochastic steady state, and projection methods may be challenging for high-dimensional problems or problems with strong nonlinearity.[5] Moreover, neither perturbation nor projection can solve problems with kinks in general, except the OccBin method (Guerrieri & Iacoviello 2015), which can solve some low-dimensional problems with occasionally binding constraints. Fernandez-Villaverde et al. (2016) provide a detailed discussion about perturbation and projection. Recently, Levintal (2018) proposes an efficient Taylor projection method to solve DSGE models. The algorithm is a hybrid of perturbation and projection, and it can obtain a locally accurate solution around any point on the state space. Fernandez-Villaverde & Levintal (2018) apply the Taylor projection method to solve a DSGE model with Epstein–Zin preferences and rare disasters. Other methods include NLCEQ (Cai et al. 2017c), discussed later in this review, and simulation-based methods including GSSA (Judd et al. 2011) and EDS (Maliar & Maliar 2015), which unfortunately cannot guarantee convergence. In addition, Dynare (Adjemian et al. 2011), a MATLAB toolbox, is used for solving DSGE models, particularly for problems in macroeconomics.

6. ROBUST DECISION MAKING

Experts often provide different models, projected paths, or estimated parameter values, so policy makers have to face Knightian uncertainty, where a particular probability distribution cannot be assigned across the models, projected paths, or parameter values. A typical method to deal with problems involving Knightian uncertainty is sensitivity analysis, and uncertainty quantification is another method (see Cai et al. 2018b, Harenberg et al. 2019). But neither can provide a robust solution for decision makers.

Robust decision-making methods help decision makers who face Knightian uncertainty. The max-min method is the most well-known robust decision-making method. It tries to maximize the minimal welfare across the uncertain models, projected paths, or estimated parameter values; that is, the max-min method corresponds to the worst-case analysis. Thus, the robust decision from the max-min method is often too conservative.

Recently, a min-max regret (MMR) method, a less conservative robust decision-making method, has been applied in environmental and resource economics for policy analysis. For an unknown but true model, there is an optimal solution to achieve the maximal welfare under the model. Other models will also propose their corresponding solution. Implementing the proposed decisions from the other models in reality (i.e., the true model) gives us realized welfare. MMR defines regret to be the difference between the maximal welfare using the optimal decisions under the true model and the realized welfare using the proposed decisions under the other models, and then chooses a robust decision to minimize the maximal regret.

Iverson (2012) implements an iterative approach and applies MMR to climate policy analysis using DICE-2007 (Nordhaus 2008) under Knightian uncertainty across weights on environmental or growth objectives, climate sensitivity, and the coefficient of the damage function of a Dynamic Integrated Climate-Economy (DICE) model. Iverson (2013) uses MMR to consider a robust environmental policy decision in the face of Knightian uncertainty about the discount rate. Anthoff & Tol (2014b) also analyze MMR using the Climate Framework for Uncertainty, Negotiation and Distribution (FUND) integrated assessment model. Cai & Sanstad (2016) introduce an efficient computational method to solve MMR problems and make robust decisions over Knightian uncertainty, and apply it to the Goulder-Mathai model (Goulder & Mathai 2000) for studying

[5] We can transform nonstationary problems into stationary problems by adding some extra state variables, so that perturbation or projection methods can be applied.

carbon emissions abatement from the energy sector in the face of model uncertainty about technical change. Cai et al. (2017a) apply the efficient MMR method to study robust decisions of agricultural research and development under uncertainty in population, income, and temperature using five shared socioeconomic pathways (O'Neill et al. 2014). Cai et al. (2016a) extend the efficient MMR method to study robust decisions of agricultural research and development under ambiguity over risk of economic growth (i.e., Knightian uncertainty across probability distributions of economic growth), based on a survey of economists about economic growth in the next century (Christensen et al. 2018, Gillingham et al. 2018). The MMR methods cannot change the level of ambiguity aversion, and they have no risk aversion (except in Cai et al. 2016a). Hansen & Sargent (2008) introduce a robust control framework in the face of both risk and ambiguity (misspecification), with both risk aversion and ambiguity aversion. Athanassoglou & Xepapadeas (2012) implement the robust control framework to consider an analytical pollution control problem, and Rudik (2016) incorporates it numerically in DICE to include learning and solves his model using VFI with sparse grid approximation. Drouet et al. (2015) disentangle model uncertainty and risks to economic production due to mitigation costs, climate dynamics, and climate damages. Berger et al. (2017) apply the robust tools in Cerreia-Vioglio et al. (2013) and Marinacci (2015) to disentangle the role of preferences from the structure of model uncertainty in order to study the impact on optimal mitigation policy.

7. OTHER COMPUTATIONAL METHODS

7.1. Nonlinear Certainty Equivalent Approximation

It is often challenging to solve dynamic stochastic programming problems with high dimensions or occasionally binding constraints. Cai et al. (2017c) introduce a new computational method, NLCEQ, to solve these kinds of problems. NLCEQ can solve deterministic infinite-horizon stationary problems accurately, and stochastic infinite-horizon stationary problems with acceptable accuracy, including a social planner's problems and competitive equilibrium. It is simple for coding and naturally parallelizable, and it is also very stable, particularly for solving a social planner's problems. For example, NLCEQ can solve a stochastic multi-country optimal growth problem with up to 400 state variables using Smolyak grids and parallelism, a dynamic model of food and clean energy with a stochastic jump process, and a New Keynesian DSGE model with a zero lower bound. See Cai et al. (2017c) for a comparison between NLCEQ and perturbation, OccBin, GSSA, and EDS methods.

7.2. Approximate Dynamic Programming

Simulation-based methods [e.g., GSSA (Judd et al. 2011)] can avoid the so-called curse of dimensionality, based on the properties of Monte Carlo simulation methods. Approximate dynamic programming (ADP) (e.g., Powell 2007) is a simulation-based and nested inner–outer iteration method. It was originally designed for problems with discrete states. It starts with a given initial guess of value functions at all times, $\widehat{V}_t^0(\mathbf{x}_t)$, where $\mathbf{x}_t$ are discrete state variables; it then updates them by the outer iteration. In each outer iteration, it generates a new sample path of stochastic variables and then runs an inner forward iteration over time using the Bellman equation. That is, after the $(n-1)$-th outer iteration, we have $\widehat{V}_t^{n-1}(\mathbf{x}_t)$ for all t; then the inner iteration starts with the initial state $\mathbf{x}_0^n$ and a simulated ϵ_0^n, uses the Bellman equation and $\widehat{V}_1^{n-1}(\mathbf{x}_1)$ to compute the optimal decision $\mathbf{a}_0^n$, computes $\widehat{V}_0^n(\mathbf{x}_0^n)$ as a weighted sum of $\widehat{V}_0^{n-1}(\mathbf{x}_0^n)$ and the optimal objective value at $\mathbf{x}_0^n$, lets $\widehat{V}_0^n(\mathbf{x}_0) = \widehat{V}_0^{n-1}(\mathbf{x}_0)$ for all $\mathbf{x}_0 \neq \mathbf{x}_0^n$, and then obtains the next-period state using the transition laws $\mathbf{x}_1^n = \mathbf{f}_0(\mathbf{x}_0^n, \mathbf{a}_0^n, \epsilon_0^n)$. With $\mathbf{x}_1^n$ and a simulated ϵ_1^n, it can similarly obtain $\widehat{V}_1^n(\mathbf{x}_1)$ and $\mathbf{x}_2^n$.

This forward iteration is continued until the terminal time, and then the inner iteration obtains $\widehat{V}_t^n(\mathbf{x}_t)$ for all t. Thus, in the n-th outer iteration, $\widehat{V}_t^n(\mathbf{x}_t)$ differentiates with $\widehat{V}_t^{n-1}(\mathbf{x}_t)$ at only one visited state $\mathbf{x}_t^n$ at each time t. The inner–outer iteration process stops until the value functions converge, that is, $\widehat{V}_t^n(\mathbf{x}_t)$ and $\widehat{V}_t^{n+1}(\mathbf{x}_t)$ are sufficiently close for all t and all states $\mathbf{x}_t$. Since this method is based on Monte Carlo simulation, it requires a large number of outer iterations; otherwise many states may not be visited with enough frequency or may even never be visited, limiting the accuracy of ADP.

For problems with continuous state variables $\mathbf{x}_t$, a standard ADP needs to discretize them, but discretization makes it inaccurate for high-dimensional problems. However, ADP can also employ value function approximation $\widehat{V}_t(\mathbf{x}_t; \mathbf{B}_t^n)$, where $\mathbf{B}_t^n$ are approximation coefficients over some basis functions, so the outer iteration updates $\mathbf{B}_t^n$ instead of values at a large discretized state space. Thus, ADP can be fast in some cases, although it may be unstable in other cases. A good choice of basis functions can significantly improve the performance of ADP. Shayegh & Thomas (2015) design a two-step-ahead approximation in ADP to solve problems with continuous state variables, in which they choose utility functions of subsequent states in the next two periods as the basis functions. The two-step-ahead algorithm is then applied in Heutel et al. (2016) and its extended four-step-ahead algorithm is applied in Heutel et al. (2018) to study the impact of solar geoengineering on climate policy under uncertainty. However, it is important to check errors (e.g., Euler errors), because the accuracy of the multiple-step-ahead approximation may be limited for approximating value functions, and ADP with the multiple-step-ahead approximation cannot guarantee that it actually solves the original dynamic stochastic problems even if it converges.

7.3. Real Options Analysis

The costs and benefits of an action in a decision-making problem are often uncertain, particularly in a dynamic environment as future management and policy can respond to new information. Real options analysis can take into account uncertainty and also flexibility, so it is often used to value the flexibility in an investment project, including allowances for future deferral, abandonment, or expansion of the project (see, e.g., Brennan & Schwartz 1985, Dixit & Pindyck 1994).

The most common methods for evaluating options are Monte Carlo simulation, decision trees, and partial differential equations (PDEs). For instance, Albers et al. (1996) use a real options approach with a decision tree to discuss the impact of uncertainty and irreversibility on the valuation and management of tropical forests, assuming that there are three types of land use: preservation, an intermediate use, and development. Insley (2002) introduces a real options approach based on a PDE to model the optimal tree-harvesting decision by implementing an implicit finite difference method to discretize a linear complementarity equation for determining the value of the option in a backward iteration. Hansen et al. (2008) evaluate an annual dry-year option, under which a water agency buys the right to purchase water at a later date with a prespecified strike price, by constructing a distribution of shadow prices that reflect the economic value of water under a simulation–optimization framework. Anda et al. (2009) apply real options analysis based on Monte Carlo simulation to select a future-flexible climate policy that can be corrected in the future in response to new knowledge. Leroux et al. (2009) use a real options approach based on a PDE and its finite difference solution to find optimal levels of conservation and land development under future stochastic natural damages, with the ecological mechanism of extinction debt as an illustration. Nadolnyak et al. (2011) apply real options analysis to market entry of genetically modified crops using VFI with Chebyshev polynomial approximation. Linquiti & Vonortas (2012) formulate a Monte Carlo model with real options analysis to test adaptation strategies for defending against sea level rise due to global warming. Ryu et al. (2018) apply real options analysis with a binomial tree to study flood mitigation strategies under uncertainty in global climate change.

7.4. Solving Principal-Agent Models

Baldwin et al. (2018) apply the Mathematical Programming with Equilibrium Conditions (MPEC) method to solve a dynamic principal-agent model, where the principal decides dynamic carbon taxes and/or subsidies to maximize social welfare, and the agents maximize their respective utility functions: The representative household maximizes the present value of utilities; the final good firms, fossil fuel firms, and renewable energy firms maximize their present value of profits. MPEC approaches have also been applied in other fields of economics. For example, Su & Judd (2012) apply MPEC in structural estimation to maximize the likelihood subject to equilibrium conditions from a Bellman equation, which is a bilevel optimization problem like the principal-agent structure, and then compare it with the nested fixed-point approach (Rust 1987). Recently, a polynomial optimization approach has been introduced to solve principal-agent models (see Renner & Schmedders 2015, 2016).

8. CONTINUOUS TIME DYNAMIC PROGRAMMING PROBLEMS

Researchers often use continuous time dynamic programming for modeling. Deterministic problems can be formulated as

$$\begin{aligned} \max \quad & \int_0^T e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t), t) dt + e^{-\rho T} W(\mathbf{x}(T)) \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t), t), \end{aligned} \tag{14.}$$

where ρ is the discount rate, $\mathbf{a}(t)$ is the vector of decision variables at time t, $\dot{\mathbf{x}}(t)$ is the derivative of the state variables $\mathbf{x}$ over time t, and the terminal time T can be infinite. Under some conditions that economic problems often satisfy, the above problem can be reformulated as the following Hamilton-Jacobi-Bellman (HJB) PDE:

$$\frac{\partial V}{\partial t}(\mathbf{x}, t) - \rho V(\mathbf{x}, t) + \max_{\mathbf{a}} \left\{ \nabla V(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, \mathbf{a}, t) + u(\mathbf{x}, \mathbf{a}, t) \right\} = 0 \tag{15.}$$

subject to the terminal condition $V(\mathbf{x}, T) = W(\mathbf{x})$, where $\nabla V(\mathbf{x}, t)$ denotes the gradient vector of V on $\mathbf{x}$.

Sometimes it is easy to derive an analytical formula for the maximizer in Equation 15, so the HJB equation can be transformed to a standard PDE that can be solved by standard computational methods, such as finite element methods and finite difference methods. Pontryagin's maximum principle can also be implemented to derive a set of equations for us to solve and obtain optimal policy functions. For example, Sohngen & Mendelsohn (1998) use this method and a shooting algorithm for solving equations to find equilibrium prices and timber harvests under climate change. Sohngen & Mendelsohn (2003) later combine a continuous time global timber model (Sohngen et al. 1999) with the discrete time DICE model (Nordhaus & Boyer 2000), and implement an iteration method to calculate carbon rental rates and then solve the two models simultaneously.

When there are occasionally binding constraints in Equation 14, HJB equations may be challenging to solve, or the value function $V(\mathbf{x}, t)$ may not even be twice differentiable over the state variables $\mathbf{x}$. Cai et al. (2012) apply finite difference methods to solve a continuous time DICE problem where the emission control rate will hit its upper bound after some years. They implement explicit, implicit, or trapezoid finite difference rules to discretize the ordinary differential equation in Equation 14, and they employ corresponding numerical integration rules to replace the integration in Equation 14 with summation. They can efficiently solve their problem with

weekly time steps, a 600-year horizon, and six continuous state variables. Moreover, their method avoids the kink problems that arise from the transformation to an HJB equation.

The stochastic version of Equation 14 is

$$\begin{aligned} \max \quad & \mathbb{E}\left\{\int_0^T e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t), t)dt + e^{-\rho T} W(\mathbf{x}(T))\right\} \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t), \epsilon(t), t), \end{aligned} \tag{16.}$$

where $\epsilon(t)$ is a continuous time stochastic process (and can be multidimensional). The time discretization method used by Cai et al. (2012) can still be applied, and then we can implement NLCEQ, value function iteration, or other computational methods for discrete time problems. The model in Equation 16 can also be converted to an HJB PDE equation under some conditions [e.g., $\epsilon(t)$ is normal, log-normal, or binary]. For example, in finance, the Black-Scholes equation for pricing a derivative is derived under a number of assumptions, including the assumption that the underlying asset's price follows a geometric Brownian motion. Polasky et al. (2011) build a system of HJB equations for problems with potential regime shifts with exogenous or endogenous probabilities. Van der Ploeg & de Zeeuw (2016) use a system of HJB equations to investigate cooperative and noncooperative responses to climate change with a North-South model of the global economy in the face of stochastic tipping points of productivity. Since the expectation operator disappears in the converted HJB equation, we can again implement standard computational methods for solving PDE on the HJB equations.

The time discretization method may be time consuming if the time horizon is large and time increments used are small, but it is becoming feasible with modern computational power, as shown by Cai et al. (2012). However, if there are multiple optimal solutions, it is still challenging to find the global optimizer or all local optimizers.

In the literature of continuous time dynamic programming problems in environmental and resource economics, many problems are deterministic and stationary assuming an infinite time horizon. For such a deterministic infinite-horizon stationary problem,

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(\mathbf{x}(t), \mathbf{a}(t))dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{a}(t)), \end{aligned} \tag{17.}$$

its corresponding HJB equation becomes

$$\max_{\mathbf{a}} \left\{\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{a}) + u(\mathbf{x}, \mathbf{a})\right\} = \rho V(\mathbf{x}). \tag{18.}$$

Instead of solving the HJB equation to get a fixed point for the unknown value function V, we often form a current value Hamiltonian,

$$\mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) = \lambda^\top \mathbf{f}(\mathbf{x}, \mathbf{a}) + u(\mathbf{x}, \mathbf{a}),$$

and then implement Pontryagin's maximum principle to obtain the following modified Hamiltonian system of ordinary differential equations (ODEs):

$$\begin{cases} \dot{\lambda} = -\nabla_{\mathbf{x}} \mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) + \rho\lambda \\ 0 = \nabla_{\mathbf{a}} \mathcal{H}(\mathbf{x}, \mathbf{a}, \lambda) \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{a}), \end{cases} \tag{19.}$$

where λ is the costate vector. Together with the transversality condition and initial/boundary conditions, we can solve the system of ODEs numerically. For example, the MATLAB ODE packages (e.g., ode45, ode15s and bvp4c) or Mathematica's NDSolve routine can be applied to solve ODE problems. In the literature, this deterministic infinite-horizon stationary problem appears in the management of a dynamic ecological system such as lake eutrophication (Carpenter et al. 1999, Brock & Starrett 2003, Mäler et al. 2003, Wagener 2003) and a socioeconomic system of a lake district on fishery management (Carpenter & Brock 2004). Grimsrud & Huffaker (2006) apply singular-perturbation reduction methods to reduce the multidimensional solution space to a lower-dimensional subspace confining long-term dynamics, and then use Mathematica's NDSolve routine to solve for optimal management of pest resistance to pesticidal crops. Fenichel & Horan (2016) apply numerical function approximation and collocation methods, a type of projection method, to solve their system of ODEs, to show the importance of institutions for managing convex-concave systems with thresholds and tipping points.

When there are multiple agents in the model, it often becomes a differential game. The ODE system in Equation 19 provides a solution to the optimal management problem under a social planner's preference, but not under decentralized equilibrium. The literature often discusses two types of differential games for decentralized equilibrium: open-loop Nash equilibrium (OLNE) and feedback Nash equilibrium (FBNE). In OLNE, agents make their decisions ignoring feedback from physical processes and strategies of other economic agents, so that OLNE has no Markov properties. We can solve OLNE via a similar system of ODEs by deriving a modified Hamiltonian system of ODEs for every agent. FBNE takes into account the feedback in the model, so agents' decisions depend on both time and state (i.e., FBNE has Markov properties), and finding a numerical solution becomes more challenging. Kossioris et al. (2008) provide a more detailed discussion about OLNE and FBNE with an application to the eutrophication of lakes, and they also introduce a numerical algorithm to solve FBNE, which incorporates the ode15s solver of MATLAB. Gopalakrishnan et al. (2017) implement the bvp4c routine of MATLAB to find OLNE for the spatial beach nourishment and coastal climate adaptation of two neighboring coastal communities.

Grass (2012) uses information about the long-run behavior of the system to derive appropriate boundary conditions at infinity, and then to reformulate the conditions in a finite time setting. His numerical algorithm is exemplified by a one-dimensional fishery model, using his MATLAB package OCMat. Grass et al. (2017) apply the algorithm and OCMat to solve for the optimal management of ecosystem services with pollution, using a lake model with fast-slow dynamics, and to find Skiba manifolds and solution paths under full cooperation (i.e., under a social planner's preference) or OLNE.

9. DISCUSSION

9.1. The Curse of Dimensionality

For multidimensional problems, the "curse of dimensionality" is often an excuse to not use VFI or time iteration (see, e.g., Traeger 2014). However, whether VFI or time iteration has the curse of dimensionality depends on the methods used. If a simple product rule[6] is used, then VFI or time iteration has the curse of dimensionality, but it may not if a nonproduct rule is used. There are

[6]For example, if Chebyshev basis functions $\mathcal{T}_k(x_i)$ with $0 \le k \le n$ and $1 \le i \le d$ are used in Equation 8 for a d-dimensional state space, then the simple product rule uses all of their products, $\mathcal{T}_{\alpha_1}(x_1) \cdots \mathcal{T}_{\alpha_d}(x_d)$, as the basis functions for all $0 \le \alpha_i \le n$ and $1 \le i \le d$ in Equation 8, so that the number of terms in the approximation is $(n+1)^d$. This method is called tensor-product Chebyshev approximation.

three potential levels of the curse of dimensionality. The first two are on the state space, and the last is on the space of random variables.

The first potential level is in the choice of approximation methods for $\widehat{V}(\mathbf{x}; \mathbf{b})$ depending on the state space, as $\widehat{V}(\mathbf{x}; \mathbf{b})$ has to be computed on the objective function of the maximization problem (Equation 7). A tensor product approximation such as piecewise linear interpolation and cubic spline interpolation will suffer from this level, as their number of basis functions $\{\phi_j(\mathbf{x})\}$ (and approximation coefficients $\mathbf{b}$) in $\widehat{V}(\mathbf{x}; \mathbf{b})$ grows exponentially in the dimension of the state space. But if we use a nonproduct approximation method, such as complete Chebyshev polynomials and simplicial complete Chebyshev polynomials described in Section 9.4, or sparse-grid interpolation (Krueger & Kubler 2004, Malin et al. 2011, Judd et al. 2014, Brumm & Scheidegger 2017), then this level of the curse of dimensionality can disappear.

The second potential level is in the choice of approximation nodes on the state space. To obtain approximation coefficients, we need to provide data $\{(\mathbf{x}_i, v_i)\}$, where v_i is the objective at the optimal solution of the maximization problem (Equation 7) with current-period state $\mathbf{x}_i$. If tensor grids $\{\mathbf{x}_i\}$ in the state space are used, then VFI or time iteration suffers from this level, unless parallelism is also used (Cai et al. 2015b).

Sparse grids, e.g., Smolyak grids (Smolyak 1963), or adaptive sparse grids (Brumm & Scheidegger 2017) can also break this level of the curse of dimensionality. Note that if the first level exists then the second level will also exist, but not vice versa, as the number of approximation coefficients could be less than the number of Lagrange data (but not vice versa) to avoid overfitting.

The last potential level is in the choice of numerical integration methods. If a tensor-product integration rule is used in computing expectations or random variables are discrete, then VFI or time iteration has the curse of dimensionality on the space of random variables ϵ_t.[7] However, when random variables are continuous, this level of the curse of dimensionality can be broken by monomial quadrature rules (Stroud 1971, Judd 1998), sparse grid integration (Gerstner & Griebel 1999, Heiss & Winschel 2008), or Monte Carlo integration methods. Monte Carlo integration methods have to use a large number of simulated points as they only have $O(1/\sqrt{N})$ accuracy with N simulated points, while the optimization solver often needs six-digit accuracy, or even higher accuracy for problems with flat objective functions in the maximization problem (Equation 7). Thus, in practice, numerical quadrature rules are often more efficient for problems with continuous random variables (Skrainka & Judd 2011). In addition, even if random variables are discrete, it is still possible to break the curse of dimensionality at this level. For example, Doraszelski & Judd (2012) avoid the curse of dimensionality in discrete time dynamic stochastic games by transforming it into a continuous time problem.

Thus, we see that all potential levels of the curse of dimensionality may not exist if we choose efficient methods in VFI or time iteration.

9.2. Boundedness

The dismal theorem of Weitzman (2009) shows that the risk premium can be infinite for unboundedly distributed uncertainties. Costello et al. (2010) use a truncation method to get bounded uncertainty and obtain a finite risk premium. Thus, numerical solutions with truncation of

[7] We often let the expectations operate on the space of next states, as the next states are random due to the randomness of ϵ_t. But in many cases the number of random variables ϵ_t in the same period t is smaller than the dimensionality of state space; for example, we may consider only one systematic shock affecting all agents, so it is beneficial to use ϵ_t. Even if the number of random variables ϵ_t is larger than the dimensionality of state space, it is often hard to construct the joint distribution of next state variables from a given joint distribution of ϵ_t.

unbounded distributions could be qualitatively inconsistent with theoretical results without truncation. However, in the literature, researchers often do not consider this inconsistency issue when they solve dynamic stochastic programming problems with an unbounded distribution. For example, a normal or log-normal distribution is often assumed for Bayesian learning when deriving Bayes' updating rules, and then researchers use a truncation method (or a bounded quadrature or simulation rule) to estimate the integration in the objective function of the maximization problem in the Bellman equation (Equation 6).

Recently, to avoid the inconsistency between theory and numerical implementation, Cai et al. (2017b) replaced a continuous long-run risk stochastic process with a two-dimensional dense Markov chain in their Dynamic Stochastic Integration of Climate and Economy (DSICE) model and then solved it numerically using VFI. Cai & Judd (2015) define a bounded distribution that is close to normal and then implement it in their model and numerical methods using Hermite information. In fact, it is often reasonable to assume bounded distributions instead of unbounded distributions in environmental and resource economics. For example, the climate sensitivity parameter is considered to be positive and less than ten (IPCC 2007, 2013).

9.3. Monte Carlo Techniques

The previous discussion assumes stochastic processes, but there are also uncertain parameters that are constant but unknown across time. These parameter values are often estimated from an econometric analysis, so we may know their distributions. Thus, if a distribution can be assigned to an uncertain parameter, then we can solve it using expected welfare maximization, which mimics the expected cost minimization method described in Cai & Sanstad (2016). Some researchers use a Monte Carlo method, which obtains an optimal policy by solving a deterministic welfare maximization problem for each sampled realization of the uncertain parameters under the distributions and then averages over the policies as an approximate solution in the face of the uncertainty. For example, New & Hulme (2000), Nordhaus (2008), Ackerman et al. (2010), and Anthoff & Tol (2013) implement this Monte Carlo method to analyze the impact of uncertainty on climate policy. While this Monte Carlo analysis can be helpful in some cases, it does not solve the real problem of a decision maker facing the parameter uncertainty, and it may even lead to the opposite sign for the effect of uncertainty (Crost & Traeger 2013).[8] Here I use a simple portfolio optimization problem with one stock and one bond to illustrate this point. Assume that the bond has a riskless 3% return, and the stock's return parameter is uncertain but we know that it has a 70% probability of being larger than 3%. The Monte Carlo method will always find it optimal to invest around 70% of wealth on the stock and the remaining 30% on the bond, no matter which risk aversion coefficient or utility function is used in the objective function.

9.4. Approximation

Value function approximation is used in the objective function of the maximization problem (Equation 7). For example, Chebyshev basis functions are used for Chebyshev approximation. Piecewise linear interpolation and cubic spline interpolation are also often used in the literature (see, e.g., Judd 1998 and Miranda & Fackler 2002). Kelly & Kolstad (1999) apply neural networks for approximation. Sometimes value functions have special properties, so special basis functions can be chosen. For example, Hwang (2017) presents a log-linearization method that approximates value functions by a linear combination of the logarithm of state variables, but this method only

[8] Lemoine & Rudik (2017) also discuss this Monte Carlo method in detail.

works for problems with value functions that can be approximated well on a reasonable domain by the log-linearization, while most problems do not have this property.[9]

For multidimensional approximation, an efficient approximation method is complete Chebyshev approximation (see, e.g., Cai & Judd 2010, 2014) or simplicial complete Chebyshev approximation introduced by Cai et al. (2018b), as both methods have no curse of dimensionality. Moreover, Chebyshev coefficients can be efficiently computed by the Chebyshev regression algorithm (see, e.g., Judd 1998 and Cai et al. 2018b) if we choose Chebyshev nodes as approximation nodes. Readers are referred to the **Supplemental Appendix** for a detailed discussion.

Supplemental Material >

Since most approximation methods are defined on hyperrectangles, we often have to truncate an unbounded or too-large state space into a bounded hyperrectangle $[\mathbf{x}_{\min}, \mathbf{x}_{\max}]$ that is wide enough to contain all necessary states. If the hyperrectangle is too narrow, then it may lead to a bad approximation for points outside the hyperrectangle as extrapolation often does not work well. Thus, solutions using VFI with narrow hyperrectangles may not be reliable. If the hyperrectangle is too wide, then it requires more approximation nodes and a higher degree of approximation to achieve the necessary accuracy, so the problem may be too time consuming or even infeasible to run with a modern computer. Since an initial state $\mathbf{x}_0$ is given in dynamic stochastic programming problems (Equation 5), we can choose a very narrow initial state space and then expand it gradually in the next periods to contain all states originated from any states in the initial state space and reasonable decisions. Cai & Judd (2012) use time-dependent state spaces to solve a simple dynamic portfolio example, where the next period's state space is chosen to contain all possible states transited from any states in current-period state space, defined as an interval of wealth.

Cai et al. (2015a, 2016b, 2017b, 2018b) and Lontzek et al. (2015) choose a series of approximation domains by setting consumption-output ratios and emission control rates in reasonable ranges, e.g., the optimal states and decisions of their corresponding deterministic dynamic programming models should be well inside the approximation domains and the ranges of decisions at each time; and simulated paths of states for the stochastic model should be well within the approximation domains. They then efficiently solve large-dimensional (from 7 to 15 dimensions) dynamic stochastic IAMs based on the framework of DSICE, in which its corresponding deterministic IAM is the annual analog of the DICE model (Nordhaus 2008). The largest example in Cai et al. (2017b) has six continuous state variables (corresponding to DICE) and three dense discrete stochastic state variables. Its horizon is 600 years and it uses annual time steps. It is solved in less than eight hours using 110,688 cores in parallel on the Blue Waters supercomputer. The smallest example in Cai et al. (2015a) has six continuous state variables (corresponding to DICE) and one binary stochastic state variable indicating whether a tipping event happens or not, and it has 600 annual time steps, but it took only minutes to get an accurate solution on a laptop. Cai et al. (2016b) solve DSICE with five interacting tipping elements in the climate system, which has ten continuous state variables and five binary stochastic state variables. Although the problems in Cai et al. (2016b) have more state variables than in Cai et al. (2017b), they can use narrower approximation domains and then lower-degree Chebyshev approximation methods as well as a much smaller number of discrete states, so they can be solved in about three hours using 10,560 cores of the Blue Waters supercomputer. All of these problems have 0.1–1% estimated errors for policy functions and 0.01–0.1% for the value functions.

[9]A linearization or log-linearization method may work locally on a narrow domain, but it is often not enough to obtain a globally accurate solution; see Cai et al. (2017c) for more discussion. In fact, from a numerical result of Hwang (2017), we can see that his solution from the log-linearization method may lead to an error of around 100% in carbon emission control compared to the solution from the optimal control method, which can be treated as the true solution.

The stationary problem (Equation 13) requires only one hyperrectangle as its approximation domain for all iterations until VFI converges. Some researchers transform the nonstationary problems (Equation 6) into the problem

$$V(\mathbf{x},\tau) = \max_{\mathbf{a}_\tau \in \mathcal{D}_\tau(\mathbf{x})} \quad u_\tau(\mathbf{x},\mathbf{a}_\tau) + \beta \mathbb{E}_\tau \left\{ V(\mathbf{x}_+,\tau_+) \right\} \qquad 20.$$
$$\text{s.t.} \quad \mathbf{x}_+ = \mathbf{f}_\tau(\mathbf{x},\mathbf{a}_\tau,\epsilon_\tau)$$
$$\tau_+ = g(\tau)$$

by adding τ as an extra continuous state variable in the value function V, where τ is bounded and has a one-to-one monotonic map to time t. For example, Lemoine & Traeger (2014) apply this trick to solve a four-dimensional dynamic stochastic IAM based on a reduced system of DICE. However, this trick increases one dimension and also has to expand its approximation domain significantly because it has to contain the minimal and maximal states along time, while states could increase significantly along time. Thus, the approximation domain would be significantly wider than the largest domain using the time-dependent state spaces. Therefore, this trick makes VFI much more time consuming. In addition, Lemoine & Traeger (2014) implement tensor-product Chebyshev approximation and MATLAB. These reasons explain why their run took days using a laptop.

9.5. Stopping Criterion

Infinite-horizon stationary dynamic programming problems (as in Equation 13) require a stopping criterion for value function iteration or time iteration. Inattention to the choice of stopping criteria may lead to large numerical errors, even if the value function has a proper magnitude and the discount factor β is not very close to 1. For example, Lemoine & Traeger (2014) use the stopping criterion

$$\max_{j=1,\ldots,N} \left| b_{k,j} - b_{k+1,j} \right| \leq 10^{-4},$$

where $\{b_{i,j} : j = 1,\ldots,N\}$ are value function approximation coefficients of tensor-product Chebyshev polynomials at the k-th iteration. However,

$$\left| \widehat{V}_k(\mathbf{x};\mathbf{b}_k) - \widehat{V}_{k+1}(\mathbf{x};\mathbf{b}_{k+1}) \right| = \left| \sum_{j=1}^{N} b_{k,j}\phi_j(\mathbf{x}) - \sum_{j=1}^{N} b_{k+1,j}\phi_j(\mathbf{x}) \right|$$
$$\leq N \max_{j=1,\ldots,N} \left| b_{k,j} - b_{k+1,j} \right| \leq 10^{-4} N,$$

where $\phi_j(\mathbf{x})$ are Chebyshev basis functions with 1 as the maximal value. Thus, the upper bound of $|\widehat{V}_i(\mathbf{x};\mathbf{b}_i) - \widehat{V}_{i+1}(\mathbf{x};\mathbf{b}_{i+1})|$ is $10^{-4}N$, not 10^{-4}. If N is huge, then the errors could be huge too. Lemoine & Traeger (2014) use $N = 10{,}000$, so the upper bound of $|\widehat{V}_i(\mathbf{x};\mathbf{b}_i) - \widehat{V}_{i+1}(\mathbf{x};\mathbf{b}_{i+1})|$ could be 1.

10. CONCLUSION

I have reviewed various state-of-the-art computational methods and their application in environmental and resource economics. Each computational method has its advantages and disadvantages.

For example, VFI and time iteration are quite general, but they require more complicated computational techniques such as efficient approximation, appropriate approximation domains, efficient numerical integration, and/or a suitable stopping criterion. NLCEQ is relatively simple and robust and can have accuracy within two or three digits, but it is limited for obtaining higher accuracy for stochastic problems. Researchers should choose a proper algorithm for their specific problem, and it is also important to verify and check the accuracy of the solution, because it is often hard to guarantee that the numerical solution found via computational methods is actually close to the true solution (for reasons such as nonlinearity, multiplicity of local optimizers, numerical errors, or bugs in code). In addition, with advances in computer technology and computational methods, it becomes feasible to solve a high-dimensional dynamic stochastic programming problem even on a laptop.

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LITERATURE CITED

Ackerman F, Stanton EA, Bueno R. 2010. Fat tails, exponents, extreme uncertainty: simulating catastrophe in DICE. *Ecol. Econ.* 69:1657–65

Adjemian S, Bastani H, Karame F, Juillard M, Maih J, et al. 2011. *Dynare: reference manual version 4*. Dynare Work. Pap. 1, CEPREMAP, Paris. **https://www.dynare.org/wp-repo/dynarewp001.pdf**

Albers H, Fisher A, Hanemann W. 1996. Valuation and management of tropical forests: implications of uncertainty and irreversibility. *Environ. Resour. Econ.* 8:39–61

Anda J, Golub A, Strukova E. 2009. Economics of climate change under uncertainty: benefits of flexibility. *Energy Policy* 37:1345–55

Anthoff D, Tol R. 2013. The uncertainty about the social cost of carbon: a decomposition analysis using FUND. *Climatic Change* 117:515–30

Anthoff D, Tol R. 2014a. *The climate framework for uncertainty, negotiation and distribution (FUND)*. Tech. Descr. Version 3.9. **http://www.fund-model.org/Fund-3-9-Scientific-Documentation.pdf**

Anthoff D, Tol R. 2014b. Climate policy under fat-tailed risk: an application of FUND. *Ann. Oper. Res.* 220:223–37

Athanassoglou S, Xepapadeas A. 2012. Pollution control with uncertain stock dynamics: when, and how, to be precautious. *J. Environ. Econ. Manag.* 63:304–20

Baldwin E, Cai Y, Kuralbayeva K. 2018. *To build or not to build? Capital stocks and climate policy*. CESifo Work. Pap. 6884, CESifo Group, Munich

Bellman R. 1957. *Dynamic Programming*. Princeton, NJ: Princeton Univ. Press

Berger L, Emmerling J, Tavoni M. 2017. Managing catastrophic climate risks under model uncertainty aversion. *Manag. Sci.* 63:749–65

Bergman L. 2005. CGE modeling of environmental policy and resource management. In *Handbook of Environmental Economics*, Vol. 3, ed. KG Maler, JR Vincent, pp. 1273–306. Amsterdam: Elsevier

Bertsekas D. 2005. *Dynamic Programming and Optimal Control*, Vol. I. Nashua, NH: Athena Sci.

Bertsekas D. 2007. *Dynamic Programming and Optimal Control*, Vol. II. Nashua, NH: Athena Sci.
Brennan MJ, Schwartz ES. 1985. Evaluating natural resource investments. *J. Bus.* 58:135–57
Brock W, Starrett D. 2003. Managing systems with non-convex positive feedback. *Environ. Resour. Econ.* 26:575–602
Brumm J, Scheidegger S. 2017. Using adaptive sparse grids to solve high-dimensional dynamic models. *Econometrica* 85:1575–612
Byrd R, Nocedal J, Waltz R. 2006. KNITRO: an integrated package for nonlinear optimization. In *Large-Scale Nonlinear Optimization*, ed. G Di Pillo, M Roma, pp. 35–59. New York: Springer
Cafiero C, Bobenrieth E, Bobenrieth J, Wright BD. 2011. The empirical relevance of the competitive storage model. *J. Econom.* 162:44–54
Cafiero C, Bobenrieth E, Bobenrieth J, Wright BD. 2015. Maximum likelihood estimation of the standard commodity storage model: evidence from sugar prices. *Am. J. Agric. Econ.* 97:122–36
Cai Y, Brock W, Xepapadeas A, Judd KL. 2018a. *Climate policy under cooperation and competition between regions with spatial heat transport*. NBER Work. Pap. 24473
Cai Y, Golub AA, Hertel TW. 2017. *Developing long-run agricultural R&D policy in the face of uncertain economic growth*. Paper presented at 2017 ASSA Annual Meeting, Chicago, Jan. 6–8
Cai Y, Golub AA, Hertel TW. 2017a. Agricultural research spending must increase in light of future uncertainties. *Food Policy* 70:71–83
Cai Y, Judd KL. 2010. Stable and efficient computational methods for dynamic programming. *J. Eur. Econ. Assoc.* 8:626–34
Cai Y, Judd KL. 2012. Dynamic programming with shape-preserving rational spline hermite interpolation. *Econ. Lett.* 117:161–64
Cai Y, Judd KL. 2014. Advances in numerical dynamic programming and new applications. In *Handbook of Computational Economics*, Vol. 3, ed. K Schmedders, K Judd, pp. 479–516. Amsterdam: Elsevier
Cai Y, Judd KL. 2015. Dynamic programming with hermite approximation. *Math. Methods Oper. Res.* 81:245–67
Cai Y, Judd KL, Lenton TM, Lontzek TS, Narita D. 2015a. Environmental tipping points significantly affect the cost-benefit assessment of climate policies. *PNAS* 112:4606–11
Cai Y, Judd KL, Lontzek TS. 2012. *Continuous-time methods for integrated assessment models*. NBER Work. Pap. 18365
Cai Y, Judd KL, Lontzek TS. 2017b. *The social cost of carbon with economic and climate risks*. Econ. Work. Pap. 18113, Hoover Inst., Stanford, CA
Cai Y, Judd KL, Lontzek TS. 2018b. *Numerical dynamic programming with verification and uncertainty quantification: an application to climate policy*. Work. Pap., Ohio State Univ., Columbus, and Stanford Univ., Stanford, CA
Cai Y, Judd KL, Steinbuks J. 2017c. A nonlinear certainty equivalent approximation method for dynamic stochastic problems. *Quant. Econ.* 8:117–47
Cai Y, Judd KL, Thain G, Wright S. 2015b. Solving dynamic programming problems on computational grid. *Comput. Econ.* 45:261–84
Cai Y, Lenton TM, Lontzek TS. 2016b. Risk of multiple interacting tipping points should encourage rapid CO_2 emission reduction. *Nat. Climate Change* 6:520–25
Cai Y, Sanstad AH. 2016. Model uncertainty and energy technology policy: the example of induced technical change. *Comput. Oper. Res.* 66:362–73
Carpenter SR, Brock WA. 2004. Spatial complexity, resilience, and policy diversity: fishing on lake-rich landscapes. *Ecol. Soc.* 9(1):8
Carpenter SR, Ludwig D, Brock WA. 1999. Management of eutrophication for lakes subject to potentially irreversible change. *Ecol. Appl.* 9:751–71
Cerreia-Vioglio S, Maccheroni F, Marinacci M, Montrucchio L. 2013. Classical subjective expected utility. *PNAS* 110:6754–59
Christensen P, Gillingham K, Nordhaus W. 2018. Uncertainty in forecasts of long-run economic growth. *PNAS* 115:5409–14

Costello CJ, Neubert MG, Polasky SA, Solow AR. 2010. Bounded uncertainty and climate change economics. *PNAS* 107:8108–10
Crost B, Traeger CP. 2013. Optimal climate policy: uncertainty versus Monte Carlo. *Econ. Lett.* 120:552–58
Czyzyk J, Mesnier MP, More JJ. 1998. The NEOS server. *IEEE Comput. Sci. Eng.* 5:68–75
Daigneault AJ, Miranda MJ, Sohngen B. 2010. Optimal forest management with carbon sequestration credits and endogenous fire risk. *Land Econ.* 86:155–72
Deaton A, Laroque G. 1995. Estimating a nonlinear rational expectations commodity price model with unobservable state variables. *J. Appl. Econom.* 10:S9–S40
Deaton A, Laroque G. 1996. Competitive storage and commodity price dynamics. *J. Political Econ.* 104:896–923
Dirkse SP, Ferris MC. 1995. The Path Solver: a nonmonotone stabilization scheme for mixed complementarity problems. *Optim. Methods Software* 5:123–56
Dixit A, Pindyck R. 1994. *Investment Under Uncertainty*. Princeton, NJ: Princeton Univ. Press
Dixon PB, Jorgenson DW. 2013. *Handbook of Computable General Equilibrium Modeling*, Vol. 1. Amsterdam: Elsevier
Doraszelski U, Judd KL. 2012. Avoiding the curse of dimensionality in dynamic stochastic games. *Quant. Econ.* 3:53–93
Drouet L, Bosetti V, Tavoni M. 2015. Selection of climate policies under the uncertainties in the fifth assessment report of the IPCC. *Nat. Climate Change* 5:937–40
Drud AS. 1994. CONOPT—a large scale GRG code. *ORSA J. Comput.* 6:207–16
Epstein LG, Zin SE. 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica* 57:937–69
Farmer JD, Hepburn C, Mealy P, Teytelboym A. 2015. A third wave in the economics of climate change. *Environ. Resour. Econ.* 62:329–57
Fenichel EP, Horan RD. 2016. Tinbergen and tipping points: Could some thresholds be policy-induced? *J. Econ. Behav. Organ.* 132:137–52
Fernandez-Villaverde J, Levintal O. 2018. Solution methods for models with rare disasters. *Quant. Econ.* 9:903–44
Fernandez-Villaverde J, Rubio-Ramirez J, Schorfheide F. 2016. Solution and estimation methods for DSGE models. In *Handbook of Macroeconomics*, Vol. 2, ed. JB Taylor, H Uhlig, pp. 527–724. Amsterdam: Elsevier
Ferris MC, Munson TS. 2000. Complementarity problems in GAMS and the Path Solver. *J. Econ. Dyn. Control* 24:165–88
Fourer R, Gay D, Kernighan B. 2003. *AMPL: A Modeling Language for Mathematical Programming*. Albuquerque, NM: AMPL. 2nd ed.
Gerstner T, Griebel M. 1999. Numerical integration using sparse grids. *Numer. Algorithms* 18:209–32
Gill P, Murray W, Saunders M, Wright M. 1994. *User's guide for NPSOL 5.0: a FORTRAN package for nonlinear programming.* Tech. Rep., Stanford Univ., Stanford, CA
Gill PE, Murray W, Saunders MA. 2005. SNOPT: an SQP algorithm for large-scale constrained optimization. *SIAM Rev.* 47:99–131
Gillingham K, Nordhaus W, Anthoff D, Blanford G, Bosetti V, et al. 2018. Modeling uncertainty in integrated assessment of climate change: a multimodel comparison. *J. Assoc. Environ. Resour. Econ.* 5:791–826
Golub AA, Henderson BB, Hertel TW, Gerber PJ, Rose SK, Sohngen B. 2013. Global climate policy impacts on livestock, land use, livelihoods, and food security. *PNAS* 110:20894–99
Golub AA, Hertel T, Lee HL, Rose S, Sohngen B. 2009. The opportunity cost of land use and the global potential for greenhouse gas mitigation in agriculture and forestry. *Resour. Energy Econ.* 31:299–319
Gopalakrishnan S, McNamara D, Smith MD, Murray AB. 2017. Decentralized management hinders coastal climate adaptation: the spatial-dynamics of beach nourishment. *Environ. Resour. Econ.* 67:761–87
Goulder LH, Mathai K. 2000. Optimal CO_2 abatement in the presence of induced technological change. *J. Environ. Econ. Manag.* 39:1–38
Grass D. 2012. Numerical computation of the optimal vector field: exemplified by a fishery model. *J. Econ. Dyn. Control* 36:1626–58

Grass D, Xepapadeas A, de Zeeuw A. 2017. Optimal management of ecosystem services with pollution traps: the lake model revisited. *J. Assoc. Environ. Resour. Econ.* 4:1121–54
Grimsrud KM, Huffaker R. 2006. Solving multidimensional bioeconomic problems with singular-perturbation reduction methods: application to managing pest resistance to pesticidal crops. *J. Environ. Econ. Manag.* 51:336–53
Guerra EA, Bobenrieth E, Bobenrieth J, Cafiero C. 2015. Empirical commodity storage model: the challenge of matching data and theory. *Eur. Rev. Agric. Econ.* 42:607–23
Guerrieri L, Iacoviello M. 2015. Occbin: a toolkit for solving dynamic models with occasionally binding constraints easily. *J. Monetary Econ.* 70:22–38
Hansen K, Howitt R, Williams J. 2008. Valuing risk: options in California water markets. *Am. J. Agric. Econ.* 90:1336–42
Hansen LP, Sargent T. 2008. *Robustness*. Princeton, NJ: Princeton Univ. Press
Harenberg D, Marelli S, Sudret B, Winschel V. 2019. Uncertainty quantification and global sensitivity analysis for economic models. *Quant. Econ.* 10:1–41
Heiss F, Winschel V. 2008. Likelihood approximation by numerical integration on sparse grids. *J. Econom.* 144:62–80
Hertel T. 2013. Global applied general equilibrium analysis using the global trade analysis project framework. In *Handbook of Computable General Equilibrium Modeling*, ed. PB Dixon, DW Jorgenson, Vol. 1, pp. 815–76. Amsterdam: Elsevier
Heutel G, Moreno-Cruz J, Shayegh S. 2016. Climate tipping points and solar geoengineering. *J. Econ. Behav. Organ.* 132:19–45
Heutel G, Moreno-Cruz J, Shayegh S. 2018. Solar geoengineering, uncertainty, and the price of carbon. *J. Environ. Econ. Manag.* 87:24–41
Horridge M, Meeraus A, Pearson K, Rutherford TF. 2013. Solution software for computable general equilibrium modeling. In *Handbook of Computable General Equilibrium Modeling*, Vol. 1, ed. PB Dixon, DW Jorgenson, pp. 1131–81. Amsterdam: Elsevier
Hwang IC. 2017. A recursive method for solving a climate-economy model: value function iterations with logarithmic approximations. *Comput. Econ.* 50:95–110
Insley M. 2002. A real options approach to the valuation of a forestry investment. *J. Environ. Econ. Manag.* 44:471–92
IPCC (Intergov. Panel Clim. Change). 2007. *Climate Change 2007: The Physical Science Basis*. New York: Cambridge Univ. Press
IPCC (Intergov. Panel Clim. Change). 2013. *Climate Change 2013: The Physical Science Basis*. New York: Cambridge Univ. Press
Iverson T. 2012. Communicating trade-offs amid controversial science: decision support for climate policy. *Ecol. Econ.* 77:74–90
Iverson T. 2013. Minimax regret discounting. *J. Environ. Econ. Manag.* 66:598–608
Jensen S, Traeger C. 2014. Watch your step: optimal policy in a tipping climate. *Eur. Econ. Rev.* 69:104–25
Judd KL. 1992. Projection methods for solving aggregate growth models. *J. Econ. Theory* 58:410–52
Judd KL. 1998. *Numerical Methods in Economics*. Cambridge, MA: MIT Press
Judd KL, Guu SM. 1993. Perturbation solution methods for economic growth models. In *Economic and Financial Modeling with Mathematica*, ed. H Varian, pp. 80–103. Berlin: Springer Verlag
Judd KL, Maliar L, Maliar S. 2011. Numerically stable and accurate stochastic simulation approaches for solving dynamic economic models. *Quant. Econ.* 2:173–210
Judd KL, Maliar L, Maliar S, Valero R. 2014. Smolyak method for solving dynamic economic models: Lagrange interpolation, anisotropic grid and adaptive domain. *J. Econ. Dyn. Control* 44:92–123
Kelly DL, Kolstad CD. 1999. Bayesian learning, growth, and pollution. *J. Econ. Dyn. Control* 23:491–518
Kelly DL, Tan Z. 2015. Learning and climate feedbacks: optimal climate insurance and fat tails. *J. Environ. Econ. Manag.* 72:98–122
Kossioris G, Plexousakis M, Xepapadeas A, de Zeeuw A, Maler KG. 2008. Feedback Nash equilibria for non-linear differential games in pollution control. *J. Econ. Dyn. Control* 32:1312–31

Krueger D, Kubler F. 2004. Computing equilibrium in OLG models with stochastic production. *J. Econ. Dyn. Control* 28:1411–36
Leach AJ. 2007. The climate change learning curve. *J. Econ. Dyn. Control* 31:1728–52
Lemoine D, Rudik I. 2017. Managing climate change under uncertainty: recursive integrated assessment at an inflection point. *Annu. Rev. Resour. Econ.* 9:117–42
Lemoine D, Traeger C. 2014. Watch your step: optimal policy in a tipping climate. *Am. Econ. J. Econ. Policy* 6:137–66
Leroux AD, Martin VL, Goeschl T. 2009. Optimal conservation, extinction debt, and the augmented quasi-option value. *J. Environ. Econ. Manag.* 58:43–57
Levintal O. 2018. Taylor projection: a new solution method for dynamic general equilibrium models. *Int. Econ. Rev.* 59:1345–73
Linquiti P, Vonortas N. 2012. The value of flexibility in adapting to climate change: a real options analysis of investments in coastal defense. *Climate Change Econ.* 3:1250008
Ljungqvist L, Sargent T. 2000. *Recursive Macroeconomic Theory*. Cambridge, MA: MIT Press
Lontzek TS, Cai Y, Judd KL, Lenton TM. 2015. Stochastic integrated assessment of climate tipping points indicates the need for strict climate policy. *Nat. Climate Change* 5:441–44
Mäler KG, Xepapadeas A, de Zeeuw A. 2003. The economics of shallow lakes. *Environ. Resour. Econ.* 26:603–24
Maliar L, Maliar S. 2015. Merging simulation and projection approaches to solve high-dimensional problems with an application to a new Keynesian model. *Quant. Econ.* 6:1–47
Malin BA, Krueger D, Kubler F. 2011. Solving the multi-country real business cycle model using a Smolyak-collocation method. *J. Econ. Dyn. Control* 35:229–39
Marinacci M. 2015. Model uncertainty. *J. Eur. Econ. Assoc.* 13:1022–100
McCarl B, Meeraus A, van der Eijk P, Bussieck M, Dirkse S, Nelissen F. 2018. *McCarl expanded GAMS user guide version 25.1*. User Guide, GAMS, Fairfax, VA. **https://www.gams.com/latest/docs/gams.pdf**
Miranda MJ, Fackler PL. 2002. *Applied Computational Economics and Finance*. Cambridge, MA: MIT Press
Nadolnyak D, Miranda MJ, Sheldon I. 2011. Genetically modified crops as real options: identifying regional and country-specific difference. *Int. J. Ind. Organ.* 29:455–63
New M, Hulme M. 2000. Representing uncertainty in climate change scenarios: a Monte-Carlo approach. *Integr. Assess.* 1:203–13
Nordhaus W, Boyer J. 2000. *Warming the World: Economic Models of Global Warming*. Cambridge, MA: MIT Press
Nordhaus WD. 2008. *A Question of Balance: Weighing the Options on Global Warming Policies*. New Haven, CT: Yale Univ. Press
O'Neill BC, Kriegler E, Riahi K, Ebi KL, Hallegatte S, et al. 2014. A new scenario framework for climate change research: the concept of shared socioeconomic pathways. *Climatic Change* 122:387–400
Polasky S, de Zeeuw A, Wagener F. 2011. Optimal management with potential regime shifts. *J. Environ. Econ. Manag.* 62:229–40
Powell WB. 2007. *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. Hoboken, NJ: Wiley-Interscience
Renner P, Schmedders K. 2015. A polynomial optimization approach to principal-agent problems. *Econometrica* 83:729–69
Renner P, Schmedders K. 2016. *Dynamic principal-agent models*. Res. Pap. 16-26, Swiss Finance Inst., Geneva
Rudik I. 2016. *Optimal climate policy when damages are unknown*. Econ. Work. Pap. 16011, Iowa State Univ., Ames
Rust J. 1987. Optimal replacement of GMC bus engines: an empirical model of Harold Zurcher. *Econometrica* 55:999–1033
Rust J. 1996. Numerical dynamic programming in economics. In *Handbook of Computational Economics*, Vol. 1, ed. HM Amman, DA Kendrick, J Rust, pp. 619–729. Amsterdam: Elsevier
Rutherford TF. 1993. *MILES: a mixed inequality and nonlinear equation solver*. Work. Pap., Univ. Colorado, Boulder
Rutherford TF. 1995. Extension of GAMS for complementarity problems arising in applied economic analysis. *J. Econ. Dyn. Control* 19:1299–324

Rutherford TF. 1999. Applied general equilibrium modeling with MPSGE as a GAMS subsystem: an overview of the modeling framework and syntax. *Comput. Econ.* 14:1–46

Ryu Y, Kim YO, Seo SB, Seo IW. 2018. Application of real option analysis for planning under climate change uncertainty: a case study for evaluation of flood mitigation plans in korea. *Mitigation Adaptation Strateg. Glob. Change* 23:803–19

Shayegh S, Thomas VM. 2015. Adaptive stochastic integrated assessment modeling of optimal greenhouse gas emission reductions. *Climatic Change* 128:1–15

Skrainka BS, Judd KL. 2011. *High performance quadrature rules: how numerical integration affects a popular model of product differentiation*. Work. Pap., Inst. Fisc. Stud., London

Smolyak SA. 1963. Quadrature and interpolation formulas for tensor products of certain classes of functions. *Sov. Math. Dokl.* 4:240–43

Sohngen B, Mendelsohn R. 1998. Valuing the impact of large-scale ecological change in a market: the effect of climate change on U.S. timber. *Am. Econ. Rev.* 88:686–710

Sohngen B, Mendelsohn R. 2003. An optimal control model of forest carbon sequestration. *Am. J. Agric. Econ.* 85:448–57

Sohngen B, Mendelsohn R, Sedjo R. 1999. Forest management, conservation, and global timber markets. *Am. J. Agric. Econ.* 81:1–13

Stroud AH. 1971. *Approximate Calculation of Multiple Integrals*. Englewood Cliffs, NJ: Prentice-Hall

Su CL, Judd KL. 2012. Constrained optimization approaches to estimation of structural models. *Econometrica* 80:2213–30

Traeger C. 2014. A 4-stated DICE: quantitatively addressing uncertainty effects in climate change. *Environ. Resour. Econ.* 59:1–37

van der Ploeg F, de Zeeuw A. 2016. Non-cooperative and cooperative responses to climate catastrophes in the global economy: a north-south perspective. *Envir. Resour. Econ.* 65:519–40

Wagener FOO. 2003. Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *J. Econ. Dyn. Control* 27:1533–61

Weitzman ML. 2009. On modeling and interpreting the economics of catastrophic climate change. *Rev. Econ. Stat.* 91:1–19