Photo of Walter Gröbli from around 1880. (Library, ETH-Zürich.)

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GRÖBLI'S SOLUTION OF THE THREE-VORTEX PROBLEM

Hassan Aref

Institute of Geophysics and Planetary Physics, University of California, La Jolla, California 92093

Nicholas Rott

Department of Aeronautics and Astronautics, Stanford University, Stanford, California 94305

Hans Thomann

Eidgenössiche Technische Hochschule Zürich, Sonneggstrasse 3, CH-8092 Zürich, Switzerland

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INTRODUCTION

The passport, a large document 36 by 59 cm, was issued in Zürich on October 11, 1875. It described the 23 year old passport bearer, Walther Gröbli, as being 5′ 8″ tall, with black hair and eyebrows, blue eyes, "average size" nose and mouth, and "oval" chin. The passport states the traveler's intention of visiting Germany, France, England, and Austria.

Walther Gröbli (later in life he would spell his first name Walter) was born in Ober-Uzwil, Switzerland, on September 23, 1852. He was the third son of Isaak Gröbli, an embroidery and weaving manufacturer, and Elisabetha Gröbli (née Grob). Walter had two older brothers, Joseph Arnold and Hermann, a younger sister, and four younger brothers who died in their infancy.

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In Uzwil Isaak Gröbli ran a small weaving house based on a Jacquard loom. In 1863 he invented a mechanical "shuttle embroidering machine." For this invention he was awarded a medal of recognition at the Paris World Exposition in 1867, but the invention was not seized upon for commercialization until 1879. Three years thereafter Isaak Gröbli was able to found his own business in Gossau and achieved a level of financial independence. Walter's oldest brother, who in 1876 moved to the United States, is credited with inventing automated versions of embroidering machines. Our main concern here, however, is with the intellectual contributions of Walter Gröbli and we thus return to a description of his life.

Walter spent his first years and attended elementary school in Ober-Uzwil. He showed great talent, and his perceptive father saw it as a duty that he receive a higher education in spite of financial rigors. He completed "high school" at the Canton School of St. Gallen, and then, from 1871 to 1875, enrolled to study mathematics at the Eidgenössisches Polytechnikum in Zürich, today known as the Eidgenössische Technische Hochschule (ETH).

One of Gröbli's professors in Zürich was Heinrich Weber. Weber had started studies in 1860 in his hometown of Heidelberg. His first academic appointment, in 1869, was at the university there. The next year he moved to Zürich, where he remained until 1875. In that year Weber went to Königsberg (now Kaliningrad), where David Hilbert, the towering figure in mathematics of the late nineteenth and early twentieth century was one of his students. In 1883 Weber moved on to Berlin; in 1884 to Göttingen; in 1892 to Marburg; and in 1895 to Strassburg, where he remained until his death in 1913. Today Weber is best known for his textbook on the partial differential equations of theoretical physics, which appeared first in 1900 and had many later editions in Germany before 1914. The generation of physicists that created the upheaval in physics in the mid-twenties through the formulation of quantum theory was raised on this book. It was essentially an act of piety when Weber wrote on the title page of his book that this work is based on Riemann's ideas. He was referring to a book based on notes taken in Riemann's course (and compiled by Hattendorff), which appeared in 1869 with the last edition printed in 1883. Weber's book became known as "Riemann-Weber." In 1924, von Mises intended to continue Weber's act of piety by declaring that Frank-Mises was a later edition of Riemann-Weber! Today all three books are classics with new printings available.

The second chair for mathematics at the ETH was held (from 1869–1875) by Hermann Amandus Schwarz (1843–1921), whose predecessor was Christoffel (the "Schwarz-Christoffel" transformation originates here), and whose successor was the algebraist Georg Ferdinand Frobenius

(1849–1917). Gröbli studied mathematics with Weber and Schwarz, both of whom had great influence on his further scientific career.

Weber would refer to Gröbli as the best student he ever had. It is apparently Weber who suggested to Gröbli that he work on the problem of the motion of three vortices, a topic on which he submitted his "diploma" in the summer of 1875 at the ETH. At that time the diploma was the final degree at the ETH and no doctoral degree could be given. Weber was leaving for Königsberg, and Schwarz was leaving for Göttingen. Either or both of them suggested to Gröbli that he should further his studies in Berlin. It is possible that Weber communicated directly with Gustav Kirchhoff, who was professor at Heidelberg from 1854–1875. Hermann von Helmholtz was also in Heidelberg from 1858—the year his famous paper on vortex motion appeared—until 1871. In 1875 Gröbli headed for Berlin to study under Kirchhoff, Helmholtz, and other illustrious mathematicians.

Hermann Ludwig Ferdinand Helmholtz (1821–1894) is a major figure in science of the nineteenth century and several biographies are available. Particularly interesting is Margenau's introduction to the 1954 edition of Helmholtz' *On the Sensations of Tone* (Helmholtz 1954), which includes a bibliography of this most versatile scientist. Helmholtz may well have been preoccupied with matters other than vortex motion and fluid mechanics during the period Gröbli was a student in Berlin. The fourth (and last) edition of *Sensations of Tone* appeared in 1877, the same year that Helmholtz became Rektor of the University of Berlin (a position he held for one year).

Gustav Robert Kirchhoff (1824–1887), whose contributions to the physics of electrical networks, spectrum analysis, and the thermodynamics of radiation are at least as well known as his contributions to fluid mechanics, had just been appointed to the chair of mathematical physics in Berlin in 1875. He had previously been at the University of Berlin as *Privatdozent* for three years starting in 1847. Following a brief period at the University of Breslau (now Wroclaw, Poland) he became professor of physics at Heidelberg, a post he held from 1854 until the move to Berlin some twenty years later. Kirchhoff's major work *Vorlesungen über mathematische Physik* (Lectures on Mathematical Physics), which figures in our story, was first published in 1876.

Gröbli enrolled as a student of the Faculty of Philosophy at the Royal Friedrich-Wilhelms-Universität in Berlin for both the first semester (October 16, 1875–April 1, 1876) and the second (April 24–August 15, 1876). During the first semester he took courses in mathematical optics and hydrodynamics from Kirchhoff, and a course on Abelian functions from Weierstrass. In the second semester he was instructed in the theory of heat by Kirchhoff, in electrodynamics by Helmholtz, in Abelian functions and the theory of analytical functions by Weierstrass, and in the theory of probability by Kummer. He also attended a class offered by Helmholtz entitled "The logical principles of experimental science." Gröbli's attendance record from Berlin is available, and a page is shown in Figure 1. The classes are listed, various annotations concerning tuition follow, we have his assigned seat in the lecture hall, and the very interesting collection of signatures by his professors.

HELMHOLTZ' 1858 PAPER

Let us now examine the background and contents of Gröbli's main scientific work, his 1877 dissertation *Specielle Probleme über die Bewegung geradliniger paralleler Wirbelfäden* (Special problems on the motion of rectilinear parallel vortices). The model that Gröbli investigates had been introduced almost two decades earlier in a seminal thirty-page paper by Helmholtz entitled *Über Integrale der hydrodynamischen Gleichungen,*

Zweites Semester. Von 24. April **18**76 **bis** 15. August **18**76

Vorlesungen.	Vermerk des Quästors betreffend das Honorar.	Nummer des Platzes im Auditorio.	Eigenhändige Einzeichnung des Docenten.	Datum der Anmeldung.	Abgemeldet bei dem Docenten.	Datum der Abmeldung.
1. Prof. G. Kirchhoff Theorie d. Wärme	1 bezahlt	32	GKirchhoff	6/5 76	G Kirchhoff	[illegible]/7 76
2. Prof. Helmholtz Electrodynamik	2	24	Helmholtz	4/5 76	Helmholtz	24/7 76
3. Prof. Helmholtz Die log. Principien d. Erfahrungswiss.	3 [illegible]	38				
4. Prof. Weierstrass Ergänzungen zur Theorie d. Abel'schen F.	4 [illegible]		Weierstrass	3. 5 76	Weierstrass	24/7 76
5. Prof. Kummer Principien der Wahrscheinlichkeitsr.	5 4/5 76	93	Kummer	1/5 76	Kummer	26/7 76
6. Prof. Weierstrass Theorie d. anal. Functionen	6 bezahlt [illegible] 20/3 76	62	Weierstrass	24. 5 76	Weierstrass	24/7 76

Figure 1 A page from the 1876 University of Berlin attendance record of W. Gröbli. The courses are mentioned in the text. Note, in particular, the signatures of the instructors.

welche den Wirbelbewegungen entsprechen. An English translation, *On integrals of the hydro-dynamical equations, which express vortex-motion,* was published by P. G. Tait in Philosophical Magazine in 1867. About the translation Tait writes: This "version of one of the most important recent investigations in mathematical physics was made long ago for my own use, and does not pretend to be an exact translation. Professor Helmholtz has been kind enough to revise it; and it may therefore be accepted as representing the spirit of the original. A portion of the contents of the paper had been anticipated by Professor Stokes . . . but the discovery of the nature and motions of vortex-filaments is entirely novel, and of great importance."

The basic results of the investigation, Helmholtz's vorticity theorems, are today well known and may be found in most textbooks. One hardly needs to consult the original article for this material. However, it is interesting to read Helmholtz's motivation for studying vortical motion in the first place. Here is what he says (in Tait's translation):

> Hitherto, in integrating the hydrodynamical equations, the assumption has been made that the components of the velocity of each element of the fluid in three directions at right angles to each other are the differential coefficients, with reference to the coordinates, of a definite function which we shall call the *velocity potential.* Lagrange[1] no doubt has shown that this assumption is lawful if the motion of the fluid has been produced by, and continued under, the action of forces which have a potential; and also that the influence of moving solids which are in contact with the fluid does not affect the lawfulness of the assumption. And, since the greater number of natural forces which can be defined with mathematical strictness can be expressed as differential coefficients of a potential, by far the greater number of mathematically investigable cases of fluid-motion belong to this class in which a velocity potential exists.
>
> Yet Euler[2] has distinctly pointed out that there are cases of fluid-motion in which no velocity-potential exists, for instance, the rotation of a fluid about an axis when every element has the same angular velocity. Among the forces which can produce such motions may be named magnetic attractions acting upon a fluid conducting electric currents, and particularly friction, whether among the elements of the fluid or against fixed bodies. The effect of fluid friction has not hitherto been mathematically defined; yet it is very great, and, except in the case of indefinitely small oscillations, produces most marked differences between theory and fact. The difficulty of defining this effect, and of finding expressions for its measurement, mainly consisted in the fact that no idea had been formed of the species of motion which friction produces in fluids. Hence it appeared to me to be of importance to investigate the species of motion for which there is no velocity potential.

Towards the end of the paper Helmholtz introduces the model that today we would refer to as point vortices (in two dimensions) or parallel line

[1] Lagrange, J.-L. 1815. *Mécanique Analytique.* Vol. II, second ed., p. 304. Paris.
[2] Euler, L. 1755. *Histoire de l'Academie des Sciences de Berlin,* p. 292.

vortices (in three dimensions). He considers the problems of one and two such vortices and derives the following well-known conclusions:

> 1. If there be a single rectilinear vortex-filament of indefinitely small section in a fluid infinite in all directions perpendicular to it, the motion of an element of the fluid at finite distance from it depends only on the product, m, of the velocity of rotation and the section, not on the form of that section. The elements of the fluid revolve about it with tangential velocity $= m/\pi r$, where r is the distance from the centre of gravity of the filament. The position of the centre of gravity, the angular velocity, the area of the section, and therefore, of course, the magnitude m remain unaltered, even if the form of the indefinitely small section may alter.
>
> 2. If there be two rectilinear vortex-filaments of indefinitely small section in an unlimited fluid, each will cause the other to move in a direction perpendicular to the line joining them. Thus the length of this joining line will not be altered. They will thus turn about their common centre of gravity at constant distances from it. If the rotation be in the same direction for both (that is, of the same sign) their centre of gravity lies between them. If in opposite directions (that is, of different signs), their centre of gravity lies in the line joining them produced. And if, in addition, the product of the velocity and the section be the same for both, so that the centre of gravity is at an infinite distance, they travel forwards with equal velocity, and in parallel directions perpendicular to the line joining them.

Several of Helmholtz' contemporaries immediately seized upon the treasures in his paper. William Thompson, the later Lord Kelvin and a lifelong friend of Helmholtz, formulated a fundamentally important corollary; his well-known theorem is the starting point of a systematic presentation in most modern texts. He also became fascinated with the problem of configurations of vortices that could move without change of shape (cf Thomson 1867), which as one outlet led to Tait's early contributions to the theory of knots in topology, and as another led to a long ago discredited theory of "vortex atoms." J. J. Thomson, discoverer of the electron, would write his Adams Prize Essay of 1883 on vortex rings, including an analysis of the conditions for steady configurations to be stable, results that he then applied to Kelvin's vortex atom model. James Clerk Maxwell would later discuss the dynamics of "molecular vortices" in conjunction with his seminal work on electromagnetism and kinetic theory.

KIRCHHOFF'S LECTURES

In modern notation the equations of motion for a system of N point vortices on the unbounded plane, with vortex $\alpha = 1, \ldots, N$ situated at (x_α, y_α) and carrying circulation Γ_α, are

$$\frac{dx_\alpha}{dt} = -\frac{1}{2\pi} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{\Gamma_\beta(y_\alpha - y_\beta)}{\rho_{\alpha\beta}^2}, \tag{1a}$$

$$\frac{dy_\alpha}{dt} = \frac{1}{2\pi} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \frac{\Gamma_\beta (x_\alpha - x_\beta)}{\rho_{\alpha\beta}^2}, \tag{1b}$$

where

$$\rho_{\alpha\beta}^2 = (x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2. \tag{1c}$$

Except for the convention of working with angular velocity times area, $m_\alpha = \Gamma_\alpha/2$, rather than the circulation, Γ_α, this is precisely the form that we find in Kirchhoff's lectures. He goes on to make the important observation that the equations can be cast in Hamiltonian form, and from this to derive the integrals of the motion related to kinetic energy (minus "self energy"), linear, and angular impulse:

> These equations [i.e. the point vortex equations (1) above] may be written[3]
>
> $$m_1 \frac{dx_1}{dt} = \frac{\partial H}{\partial y_1}, \quad m_2 \frac{dx_2}{dt} = \frac{\partial H}{\partial y_2}, \ldots$$
>
> $$m_1 \frac{dy_1}{dt} = -\frac{\partial H}{\partial x_1}, \quad m_2 \frac{dy_2}{dt} = -\frac{\partial H}{\partial x_2}, \ldots \tag{2}$$
>
> $$H = -\frac{1}{\pi} \sum m_1 m_2 \log \rho_{12} \tag{3}$$
>
> where the sum in question is to be taken over all combinations of two different indices.[4]
>
> Certain integrals of the equations (1) may be found regardless of how large the number of vortices is. The value of H is unchanged when $x_1, x_2, \ldots$ or $y_1, y_2, \ldots$ are incremented by the same quantity. From this it follows that
>
> $$\sum \frac{\partial H}{\partial x_1} = 0 \quad \text{and} \quad \sum \frac{\partial H}{\partial y_1} = 0,$$
>
> i.e.
>
> $$\sum m_1 x_1 = \text{const.} \quad \text{and} \quad \sum m_1 y_1 = \text{const.} \tag{4}$$
>
> These equations tell us ... that the centroid of the vortices remains in the same place.[5]
>
> If we multiply the equations in the first row of (2) by $dy_1, dy_2, \ldots$, and those in the second by $-dx_1, -dx_2, \ldots$, and add the rows, we obtain
>
> $$dH = 0, \quad \text{i.e.} \quad H = \text{const.} \tag{5}$$
>
> One finds a fourth integral when one introduces polar in place of Cartesian coordinates by the following considerations. We have

[3] Kirchhoff uses the symbol P in place of H. We consistently call the Hamiltonian function H.

[4] In modern notation one would have written $H = -(1/\pi) \sum_{\alpha<\beta} m_\alpha m_\beta \log \rho_{\alpha\beta}$ with generic summation indices. The older notation writes a "typical term" and explains in words how the sum is to be carried out.

[5] This centroid of the system of point vortices is also often called the *center of vorticity*.

$$x_1 = \rho_1 \cos \vartheta_1, \quad x_2 = \rho_2 \cos \vartheta_2, \ldots$$
$$y_1 = \rho_1 \sin \vartheta_1, \quad y_2 = \rho_2 \sin \vartheta_2, \ldots. \tag{6}$$

The differential equations (2) are transformed by these substitutions into

$$m_1 \rho_1 \frac{d\rho_1}{dt} = \frac{\partial H}{\partial \vartheta_1}, \quad m_2 \rho_2 \frac{d\rho_2}{dt} = \frac{\partial H}{\partial \vartheta_2}, \ldots$$
$$m_1 \rho_1 \frac{d\vartheta_1}{dt} = -\frac{\partial H}{\partial \rho_1}, \quad m_2 \rho_2 \frac{d\vartheta_2}{dt} = -\frac{\partial H}{\partial \rho_2}, \ldots. \tag{7}$$

Since H, according to the definition given in (3), is unchanged when the angles ϑ_1, $\vartheta_2, \ldots$ are increased by the same amount, it follows that

$$\sum \frac{\partial H}{\partial \vartheta_1} = 0.$$

The equations in the top row of (7) then give

$$\sum m_1 \rho_1^2 = \text{const.} \tag{8}$$

We also derive a conclusion from the lower row of equations (7). If, while keeping the angles $\vartheta_1, \vartheta_2, \ldots$ unchanged, we multiply $\rho_1, \rho_2, \ldots$ by n, thus adding $\log n$ to $\log \rho_1, \log \rho_2, \ldots$, the quantities ρ_{12} will also grow by a factor n, and thus the $\log \rho_{12}$ will increase by $\log n$. From (3) it then follows that H increases by

$$-\frac{1}{\pi} \log n \sum m_1 m_2.$$

From this we get[6]

$$\sum \frac{\partial H}{\partial \log \rho_1} = -\frac{1}{\pi} \sum m_1 m_2$$

or

$$\sum \rho_1 \frac{\partial H}{\partial \rho_1} = -\frac{1}{\pi} \sum m_1 m_2$$

and, thus, from (7)

$$\sum m_1 \rho_1^2 \, d\vartheta_1 = \frac{dt}{\pi} \sum m_1 m_2. \tag{9}$$

Kirchhoff went on to discuss the problem of two interacting vortices, where he reproduced the solution already given by Helmholtz. The case of two opposite vortices translating forward by mutual induction, and the "atmosphere" carried along by such a pair, had been discussed in detail by Kelvin (see Thomson 1867).

It is natural next to investigate the motion of more than two point vortices, and it does not take long to realize that the solution will be much more complicated. The analogous question in the case of gravitating mass

[6] Note that the sums on the right hand sides are double sums, whereas those on the left are on a single index (cf footnote 4).

points—the famous three-body problem—was to lead, at the hands of Henri Poincaré and others, to the first glimpses of what we today know as chaos in dynamical systems. But the three-vortex problem does not display chaos. It belongs to the family of integrable systems. Gröbli's dissertation consisted in establishing this fact and elucidating details of the motion for several perceptively chosen triples of vortex strengths. In a sense, the three-vortex problem plays the same role in vortex dynamics as the two-body Kepler problem in the theory of gravitationally interacting mass points.

In his lectures Kirchhoff describes the solution of the three-vortex problem thus:

> If *three* vortices are given, the problem of determining their motion depends on the solution of equations and the execution of quadratures. Thus, if we introduce as the determining variables ρ_1, ρ_2, ρ_3, $\vartheta_2-\vartheta_1$, $\vartheta_3-\vartheta_1$ and ϑ_1, and if we multiply equations (4) first by $\cos\vartheta_1$, $\sin\vartheta_1$, then by $-\sin\vartheta_1$, $\cos\vartheta_1$, adding them each time, we may solve the ensuing equations and equations (5) and (8) for four of the quantities ρ_1, ρ_2, ρ_3, $\vartheta_2-\vartheta_1$, $\vartheta_3-\vartheta_1$ in terms of the fifth. Assume that the others are all expressed in terms of ρ_1. From (9) and the equation
>
> $$m_1\rho_1\,d\vartheta_1 = -\frac{\partial H}{\partial \rho_1}dt,$$
>
> that appears in the system of equations (7), one can find ϑ_1 and t in terms of ρ_1.

The 1883 edition of the Lectures contains a footnote: "Cf. Gröbli, Inaugural-Dissertation, Göttingen 1877." Indeed, while this simple counting of equations and variables shows that a reduction to quadrature can be accomplished, actually setting this up and elucidating the physical nature of the vortex trajectories is a major task, details of which can still give rise to research papers today, more than a century later. Poincaré, who considered the three-vortex problem in his *Théorie des Tourbillons* (Poincaré 1893, Section 77) was even more terse:

> We have thus determined three integrals of our differential equations. These properties of the equations allow us to integrate them by quadratures when there are just three vortex tubes.
>
> Indeed, our equations have the form of Hamilton's canonical equations, that may be integrated by quadratures when they contain $2n$ variables, and one knows n particular integrals. Thus, when there are three vortex tubes, the equations contain the six variables x_1, y_1, x_2, y_2, x_3, y_3, and we have found three particular integrals.

GRÖBLI'S DISSERTATION

The dissertation was an outgrowth of the work that had led to the diploma in Zürich. This was reformulated and new investigations were added to it. The subject matter was the motion of three vortices, the motion of four vortices assuming the existence of an axis of symmetry, and the motion of $2n$ vortices assuming the existence of n symmetry axes. We shall concentrate here on the three-vortex work, but should mention that Gröbli's results on the four-vortex problem are at least as comprehensive as those obtained many years later by Love (1894), who does not cite his work. Gröbli's results on the n-vortex problem are less detailed but overlap with the investigation by Greenhill (1878).

The introduction (Section 1) of the dissertation quickly summarizes the results of Helmholtz and Kirchhoff that we have already discussed. The very last sentence—"We shall not enter into the determination of the motion of fluid particles situated at a finite distance from the vortices"—is interesting in view of later developments: the motion of a passive particle in the flow field of three interacting vortices is, in general, nonintegrable, an idea that would have been quite foreign to Gröbli and his contemporaries.

To give a sense of the dissertation we follow the discussion in Section 2, entitled "On the motion of thrce vortices," where the strategy for integrating the equations is explained:

> We introduce the somewhat more convenient notation s_1, s_2, s_3 for the distances ρ_{23}, ρ_{31}, ρ_{12} between the three vortices. The differential equations (1) that determine the motion of the system of three vortices are in Cartesian coordinates

$$\pi\frac{dx_1}{dt} = -m_2\frac{y_1-y_2}{s_3^2} + m_3\frac{y_3-y_1}{s_2^2}$$

$$\pi\frac{dx_2}{dt} = -m_3\frac{y_2-y_3}{s_1^2} + m_1\frac{y_1-y_2}{s_3^2}$$

$$\pi\frac{dx_3}{dt} = -m_1\frac{y_3-y_1}{s_2^2} + m_2\frac{y_2-y_3}{s_1^2} \tag{10}$$

$$\pi\frac{dy_1}{dt} = m_2\frac{x_1-x_2}{s_3^2} - m_3\frac{x_3-x_1}{s_2^2}$$

$$\pi\frac{dy_2}{dt} = m_3\frac{x_2-x_3}{s_1^2} - m_1\frac{x_1-x_2}{s_3^2}$$

$$\pi\frac{dy_3}{dt} = m_1\frac{x_3-x_1}{s_2^2} - m_2\frac{x_2-x_3}{s_1^2} \tag{11}$$

> and in polar coordinates [see (6)]

$$\pi\frac{d\rho_1}{dt} = -\frac{m_2\rho_2\sin(\vartheta_1-\vartheta_2)}{s_3^2} + \frac{m_3\rho_3\sin(\vartheta_3-\vartheta_1)}{s_2^2}$$

$$\pi\frac{d\rho_2}{dt} = -\frac{m_3\rho_3\sin(\vartheta_2-\vartheta_3)}{s_1^2} + \frac{m_1\rho_1\sin(\vartheta_1-\vartheta_2)}{s_3^2}$$

$$\pi\frac{d\rho_3}{dt} = -\frac{m_1\rho_1\sin(\vartheta_3-\vartheta_1)}{s_2^2} + \frac{m_2\rho_2\sin(\vartheta_2-\vartheta_3)}{s_1^2} \tag{12}$$

$$\pi\rho_1\frac{d\vartheta_1}{dt}=m_2\frac{\rho_1-\rho_2\cos(\vartheta_1-\vartheta_2)}{s_3^2}+m_3\frac{\rho_1-\rho_3\cos(\vartheta_3-\vartheta_1)}{s_2^2}$$

$$\pi\rho_2\frac{d\vartheta_2}{dt}=m_3\frac{\rho_2-\rho_3\cos(\vartheta_2-\vartheta_3)}{s_1^2}+m_1\frac{\rho_2-\rho_1\cos(\vartheta_1-\vartheta_2)}{s_3^2}$$

$$\pi\rho_3\frac{d\vartheta_3}{dt}=m_1\frac{\rho_3-\rho_1\cos(\vartheta_3-\vartheta_1)}{s_2^2}+m_2\frac{\rho_3-\rho_2\cos(\vartheta_2-\vartheta_3)}{s_1^2}. \tag{13}$$

In these equations

$$s_1^2=(x_2-x_3)^2+(y_2-y_3)^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos(\vartheta_2-\vartheta_3)$$

$$s_2^2=(x_3-x_1)^2+(y_3-y_1)^2=\rho_3^2+\rho_1^2-2\rho_3\rho_1\cos(\vartheta_3-\vartheta_1)$$

$$s_3^2=(x_1-x_2)^2+(y_1-y_2)^2=\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\vartheta_1-\vartheta_2). \tag{14}$$

We assume next that $m_1+m_2+m_3$ is different from zero. Then, one can make the centroid of thc threc vortices the origin of coordinates, and the equations (4) expressing that the centroid remains at rest become

$$m_1x_1+m_2x_2+m_3x_3=0$$

$$m_1y_1+m_2y_2+m_3y_3=0 \tag{15}$$

or in polar coordinates

$$m_1\rho_1\cos\vartheta_1+m_2\rho_2\cos\vartheta_2+m_3\rho_3\cos\vartheta_3=0$$

$$m_1\rho_1\sin\vartheta_1+m_2\rho_2\sin\vartheta_2+m_3\rho_3\sin\vartheta_3=0. \tag{16}$$

We shall write the third and fourth of the general integrals (8), (3) in the following form

$$m_1\rho_1^2+m_2\rho_2^2+m_3\rho_3^2=C' \tag{17}$$

$$\frac{1}{m_1}\log s_1+\frac{1}{m_2}\log s_2+\frac{1}{m_3}\log s_3=C. \tag{18}$$

We multiply the first of equations (16) by sin ϑ_1, sin ϑ_2, sin ϑ_3 and the second by $-\cos\vartheta_1$, $-\cos\vartheta_2$, $-\cos\vartheta_3$, adding each time. In this way we obtain three equations that are most simply written as

$$\frac{\sin(\vartheta_2-\vartheta_3)}{m_1\rho_1}=\frac{\sin(\vartheta_3-\vartheta_1)}{m_2\rho_2}=\frac{\sin(\vartheta_1-\vartheta_2)}{m_3\rho_3}. \tag{19}$$

Furthermore, if in equations (16) the first term is transferred to the right hand side, and both sides are then squared, one obtains the first of the following equations from which the two next arise by cyclical permutation of indices 1, 2, 3, viz

$$\cos(\vartheta_2-\vartheta_3)=\frac{m_1^2\rho_1^2-m_2^2\rho_2^2-m_3^2\rho_3^2}{2m_2m_3\rho_2\rho_3}$$

$$\cos(\vartheta_3-\vartheta_1)=\frac{-m_1^2\rho_1^2+m_2^2\rho_2^2-m_3^2\rho_3^2}{2m_3m_1\rho_3\rho_1}$$

$$\cos(\vartheta_1-\vartheta_2)=\frac{-m_1^2\rho_1^2-m_2^2\rho_2^2+m_3^2\rho_3^2}{2m_1m_2\rho_1\rho_2}. \tag{20}$$

We substitute these expressions for $\cos(\vartheta_2-\vartheta_3)$, $\cos(\vartheta_3-\vartheta_1)$, $\cos(\vartheta_1-\vartheta_2)$ in (14), and by simultaneously using (17) arrive at the following formulae

$$
\begin{aligned}
m_2m_3s_1^2 &= (m_2+m_3)C'-m_1(m_1+m_2+m_3)\rho_1^2\\
m_3m_1s_2^2 &= (m_3+m_1)C'-m_2(m_1+m_2+m_3)\rho_2^2\\
m_1m_2s_3^2 &= (m_1+m_2)C'-m_3(m_1+m_2+m_3)\rho_3^2.
\end{aligned}
\tag{21}
$$

We introduce a new constant C'', related to C' by the equation

$$m_1m_2m_3C'' = (m_1+m_2+m_3)C'. \tag{22}$$

From the previous equations it then follows that

$$\frac{s_1^2}{m_1}+\frac{s_2^2}{m_2}+\frac{s_3^2}{m_3} = C''. \tag{23}$$

Equations (18) and (23) provide two integrals of the differential equations of our problem in which the sides of the triangle of vortices appear. This suggests that it should be relatively easy to set up three differential equations, containing only the time and the sides s_1, s_2, s_3 of the triangle, from which the instantaneous triangle shape may be determined. To derive these equations we subtract the third equation in both (10) and (11) from the second, multiply the first of the equations that thus arises by x_2-x_3, and the second by y_2-y_3, and add. In this way we get

$$\frac{\pi}{2}\frac{d(s_1^2)}{dt} = m_1\frac{s_2^2-s_3^2}{s_2^2s_3^2}\{y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)\}.$$

The quantity in curly brackets on the right hand side of this equation represents twice the positive or negative area of the triangle depending on whether the vortices 1, 2, 3 appear in the clockwise or counter-clockwise sense in the plane. As is well known this area may be expressed in terms of the sides of the triangle. If we denote the above expression by $2J$, we obtain the first of the equations (24) below, and from this one the other two follow by cyclic permutation of indices 1, 2, 3. Thus,

$$
\begin{aligned}
\frac{d(s_1^2)}{dt} &= \frac{m_1}{\pi}4J\frac{s_2^2-s_3^2}{s_2^2s_3^2}\\
\frac{d(s_2^2)}{dt} &= \frac{m_2}{\pi}4J\frac{s_3^2-s_1^2}{s_3^2s_1^2}\\
\frac{d(s_3^2)}{dt} &= \frac{m_3}{\pi}4J\frac{s_1^2-s_2^2}{s_1^2s_2^2}
\end{aligned}
\tag{24}
$$

where J is given by the equation[7]

$$16J^2 = 2s_2^2s_3^2+2s_3^2s_1^2+2s_1^2s_2^2-s_1^4-s_2^4-s_3^4. \tag{25}$$

The already known integrals (18) and (23) arise if equations (24) are divided through by m_1, m_2, m_3, respectively, or by $m_1s_1^2$, $m_2s_2^2$, $m_3s_3^2$, respectively, and added each time.

To solve the problem of finding the instantaneous shape of the triangle now only requires elimination of variables and quadrature.

The motion is entirely determined if we have one additional equation in which one

[7] This is just *Heron's formula* for the area of the triangle in terms of the lengths of its sides.

or more coordinates and the time appear. Using (20) and (21) equations (13) may be transformed so that apart from one of the derivatives

$$\frac{d\vartheta_1}{dt}, \quad \frac{d\vartheta_2}{dt}, \quad \frac{d\vartheta_3}{dt}$$

they only contain the sides s_1, s_2, s_3. Indeed, we obtain the following system of equations

$$2\pi(m_1+m_2+m_3)\rho_1^2 s_2^2 s_3^2 \frac{d\vartheta_1}{dt} = m_2 m_3\{(s_2^2-s_3^2)^2 - s_1^2(s_2^2+s_3^2)\} + 2(m_2+m_3)^2 s_2^2 s_3^2$$

$$2\pi(m_1+m_2+m_3)\rho_2^2 s_3^2 s_1^2 \frac{d\vartheta_2}{dt} = m_3 m_1\{(s_3^2-s_1^2)^2 - s_2^2(s_3^2+s_1^2)\} + 2(m_3+m_1)^2 s_3^2 s_1^2$$

$$2\pi(m_1+m_2+m_3)\rho_3^2 s_1^2 s_2^2 \frac{d\vartheta_3}{dt} = m_1 m_2\{(s_1^2-s_2^2)^2 - s_3^2(s_1^2+s_2^2)\} + 2(m_1+m_2)^2 s_1^2 s_2^2 \qquad (26)$$

in which ρ_1, ρ_2, ρ_3 are related to the quantities s by equations (21), and have only been retained here for ease of writing. These differential equations are only valid when we assume that the centroid is at the origin of coordinates.

The problem at hand may now be solved in the following way. From equations (17), (18), (20), (21) and (23) the nine quantities

$$\begin{array}{ccc} s_1, & s_2, & s_3 \\ \rho_1, & \rho_2, & \rho_3 \\ \cos(\vartheta_2-\vartheta_3), & \cos(\vartheta_3-\vartheta_1), & \cos(\vartheta_1-\vartheta_2) \end{array}$$

can all be determined as functions of a single variable τ. If these expressions are substituted in any of equations (12) or (24), one obtains t as a function of τ by quadrature, and thus τ as a function of t. With the help of (13) or (26) one now also obtains the quantities ϑ_1, ϑ_2, ϑ_3 as functions of the time. . . .

In the preceding we have assumed that the centroid of the vortices and the origin of coordinates coincide. This assumption is no longer permissible if $m_1+m_2+m_3 = 0$, since the centroid is then at infinity. In this case it is best to calculate in Cartesian coordinates. One of the axes, for example the x-axis, may be chosen to give the direction to the centroid, so that instead of equations (15), we have

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = \text{const.}$$

$$m_1 y_1 + m_2 y_2 + m_3 y_3 = 0.$$

By a suitable choice of the origin of coordinates we may achieve that the constant C' in (17) vanishes. . . . If the problem is just to find the shape of the triangle, one can use equations (24), which are valid for all values of the strengths m, since these equations are independent of any choice of coordinate system.

Thus concludes Section 2 of Gröbli's dissertation. A few sentences outlining particular cases that he later discusses in detail have been omitted. The case of vanishing total circulation receives little attention in the dissertation. From a physical point of view this is somewhat unfortunate. The constraint of vanishing total circulation is a most important one in view of the divergence-free nature of a full three-dimensional vorticity field. If the point vortices in two dimensions are thought of as intersections of a three-dimensional vortex tube with a plane, it follows that the sum of circulations must vanish. Fortunately, this particular case is one of the simplest, and a direct solution is immediately possible (Rott 1989, Aref 1989). Furthermore, for the four-vortex problem an interesting class of integrable cases occurs for systems with vanishing total circulation (Eckhardt 1988, Rott 1990) so that in this context the constraint is again significant.

Analyses of Special Cases

In Sections 3–5 of the dissertation Gröbli considers the case $m_1 = m_2 = -m_3$. This is a very interesting special case, portions of which can be recast as a scattering problem in which a pair consisting of two opposite vortices (1 and 3, say) impinges on a single "target" vortex, and the entire problem can be solved in terms of elliptic functions. Gröbli does this and identifies two types of motion: one in which vortices 1 and 3 stay together, negotiating vortex 2, and then departing for infinity; another in which 3 leaves 1 behind during the interaction, and pairs up with 2. At the cross-over points between these two we find a separatrix-type motion in which all three vortices become trapped in a (rectilinear or equilateral triangle) configuration that rotates as a rigid body (Figure 2*a*). This case of two positive and one negative vortex, all with the same magnitude of the circulation, is historically interesting because it was mentioned (without giving any analysis) by the Russian aerodynamicist Nikolay Yegorovich Zhukovsky (1847–1921) in his lecture given on the occasion of Helmholtz's seventieth birthday. It is amusing to compare the illustration in Gröbli's dissertation with that given by Zhukovsky and with the results of a current computation (Figure 2*b–d*). The undulatory motion of the negative vortex was exaggerated in the early plots.

In Section 6 Gröbli considers the symmetrical case of identical vortices $m_1 = m_2 = m_3$. Again a complete analysis is performed, this time leading to hyperelliptic functions. An intuitively appealing geometrical construction is advanced:

> The equations (17) and (18) may be written
>
> $$\rho_1^2+\rho_2^2+\rho_3^2 = 1 \tag{27}$$
>
> $$s_1 s_2 s_3 = \lambda \tag{28}$$
>
> if the unit of length is suitably chosen. Hcrc λ is a positive constant that for the time being is otherwise arbitrary.
>
> Equations (21) that establish the connection between the quantities ρ and s become
>
> $$s_1^2 = 2-3\rho_1^2, \quad s_2^2 = 2-3\rho_2^2, \quad s_3^2 = 2-3\rho_3^2. \tag{29}$$
>
> If we add these three equations, we get in view of (27) that

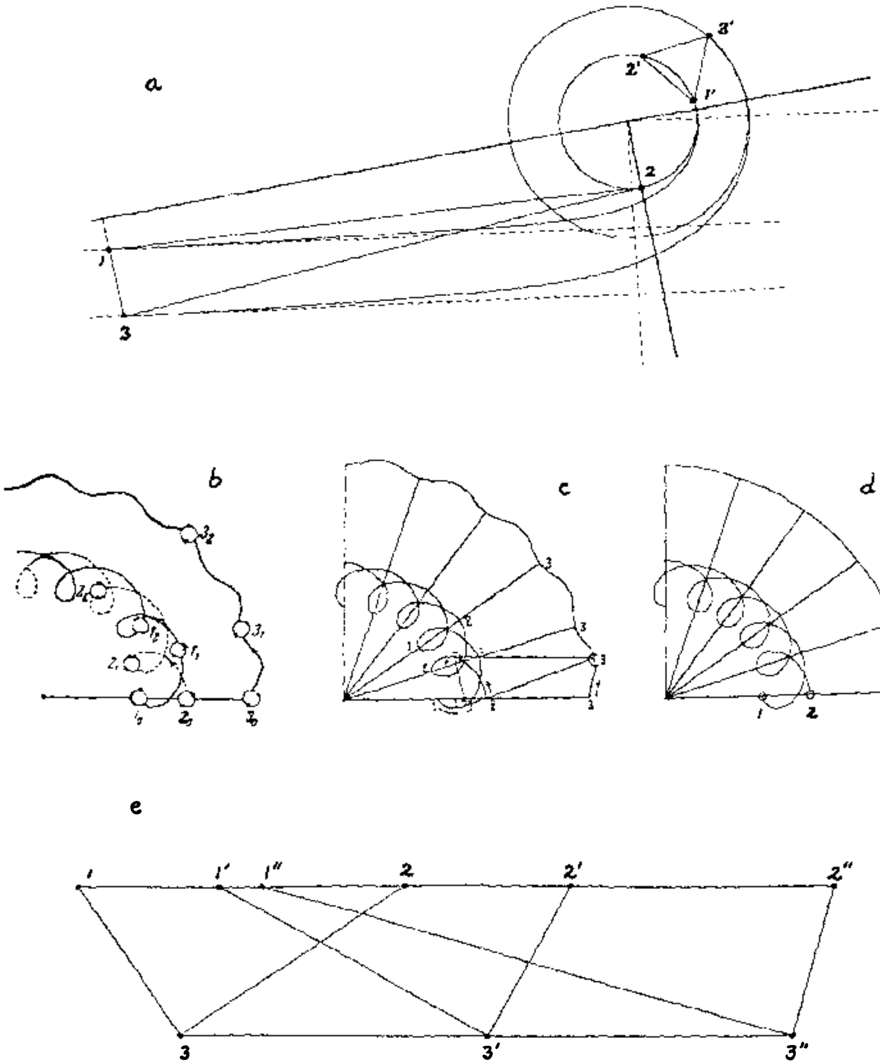

Figure 2 Examples of three-vortex motion for the case $m_1 = m_2 = -m_3$ analyzed by Gröbli in his dissertation. In (*a*) vortices *1* and *3* are paired initially and approach vortex *2*. For this particular case, where as $t \to -\infty$ vortex *2* is on the bisector of segment *13*, the configuration settles into a uniformly rotating, equilateral triangle as $t \to +\infty$ ("separatrix" motion). Panels (*b*), (*c*), and (*d*) depict the same motion; (*b*) as plotted in Zhukovsky's paper, (*c*) as plotted in Gröbli's dissertation, and (*d*) as computed in a recent simulation. The undulations in the path of the outer, negative vortex have been exaggerated in the older works. Panel (*e*) shows a case discussed by Gröbli (but not illustrated by him) in which all three vortices propagate along parallel, straight lines.

$$s_1^2+s_2^2+s_3^2 = 3. \tag{30}$$

Using equations (28) and (30) we may readily gain an overview of the motion at least so far as the shape of the triangle is concerned, and we may establish a classification into different possible cases. Let us imagine s_1, s_2, s_3 as cartesian coordinates of a point in space. Then (30) defines a sphere of radius $\sqrt{3}$, and (28) describes a surface of third order, that has the coordinate planes as asymptotes, and intersects planes parallel to the coordinate planes in rectangular hyperbolae. Since the quantities s_1, s_2, s_3 are positive, we may limit ourselves to the first octant. The planes

$$s_2 = s_3, \quad s_3 = s_1, \quad s_1 = s_2$$

are symmetry planes of the sphere, as well as for the surface of third order, and, thus, also for the oval curve of intersection between the two. Every point of this curve corresponds to a certain shape of the triangle of the three vortices. The oval has a highest and a lowest point corresponding to the two extreme values between which s_3 must lie. Of course, s_1 and s_2 must be within these limits as well. Since the plane $s_1 = s_2$ is a symmetry plane, s_3 surely attains its maximum and minimum for $s_1 = s_2$. Other maxima and minima do not occur. If we set $s_1 = s_2$, we obtain for s_3 a cubic equation

$$s_3^3-3s_3+2\lambda = 0. \tag{31}$$

One root of (31) is always real, but of no use since it is negative. The two other roots give, respectively, the maximum and minimum of s_3, and may be real and different, or real and equal, or imaginary, depending on whether the two surfaces intersect, touch or do not intersect each other. The surfaces touch when $\lambda = 1$. The curve of intersection then reduces to the point

$$s_1 = s_2 = s_3 = 1; \tag{32}$$

the three vortices form an equilateral triangle with constant sides. From (29)

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = \tfrac{1}{3}.$$

Denoting the common value of m_1, m_2, m_3 by m, we get from equations (26) that

$$\frac{d\vartheta_1}{dt} = \frac{d\vartheta_2}{dt} = \frac{d\vartheta_3}{dt} = \frac{3m}{\pi}. \tag{33}$$

Thus, the triangle of vortices rotates about its midpoint with a constant angular velocity.

Summarizing the above arguments we have the following results. The sides s_1, s_2, s_3 of the triangle of vortices must satisfy (28) and (30). In the former λ is a constant between 0 and 1.[8] Due to these constraints each side can only vary between finite limits, that are the same for all three sides, and are determined as the positive roots of the equation

$$s^3-3s+2\lambda = 0.$$

If one side has taken on an extremum value, the triangle is isosceles.

There is still one further observation to be made. Since the vortices form a triangle, the sum of two of the sides must be larger than the third. This condition is certainly fulfilled when one of the sides, say s_3, is a minimum: Since the smaller root of the above equation is less than 1, the corresponding values of s_1 and s_2 are greater than 1 by equation (30), thus $s_1+s_2 > s_3$. On the other hand, if s_3 is a maximum, it is larger than

[8] Although Gröbli does not mention it, this follows by comparing (28) and (30) using the inequality between geometric and arithmetic mean.

1, thus s_1 and s_2 are less than 1, and it depends entirely on the value of λ whether s_1+s_2 is greater or less than s_3. The limiting case, $s_1+s_2 = s_3$, arises for $s_3 = \sqrt{2}$. The cubic equation gives $1/\sqrt{2}$ as the corresponding value of λ. In one limit, then, the three vortices are on a line, one at the midpoint of the two others. The three equations

$$s_2+s_3 = s_1, \quad s_3+s_1 = s_2, \quad s_1+s_2 = s_3$$

represent the planes through the bisectors of the angles between the coordinate axes. These planes intersect the sphere in an equilateral, spherical triangle. The three cases $\lambda^2 > \frac{1}{2}$, $\lambda^2 = \frac{1}{2}$ and $\lambda^2 < \frac{1}{2}$ are distinguished by the oval (mentioned previously) being enclosed by the triangle, or touching it, or intersecting it. In the last case only one isosceles triangle shape can occur. The other limiting case is a degenerate one in which the three vortices are on a line, but not with one at the midpoint of the other two.

The case of identical vortices, in particular the stability of the equilateral triangle configuration and the instability of the rectilinear configuration, was discussed by Kelvin (see Thomson 1878) in a brief note commenting on experiments using floating magnets by Mayer (1878a,b). The geometrical construction was rediscovered almost word for word by Novikov (1975) a century later!

In Section 7 Gröbli solves the least interesting of the three special cases chosen for discussion, $m_1 = 2m_2 = -2m_3$.

Geometrically Constrained Motions

The next five sections of the dissertation discuss three-vortex motions in which some simple geometrical feature of the vortex triangle remains invariant. Thus, "rigid motions" in which the shape and size of the vortex triangle persists are treated in Section 9. In Section 10 "self-similar motions" in which the shape of the vortex triangle, but not its size, remains invariant are addressed. In Section 11 the triangle is assumed to remain equilateral. Finally, Section 12 considers cases where the three vortices move along parallel lines.

The most interesting of these is probably the case of self-similar motions where the vortices move along logarithmic spirals and can collapse to a point (which must be the centroid) in a finite time. Two conditions are necessary for self-similar motion to occur: one is that the harmonic mean of the circulations vanish; the other is that the constant C'', Equation (23), vanish. We shall not enter into details of the analysis but simply show (Figure 3) Gröbli's illustration of this type of motion and a construction, given a century later, of how to find initial conditions for collapse to occur (Aref 1979).

The special case of motion along parallel lines for three vortices with $m_1 = m_2 = -m_3$ is probably the simplest solution that can be found of the three-vortex equations in which all mutual distances change in time. This motion is shown in Figure 2*e*.

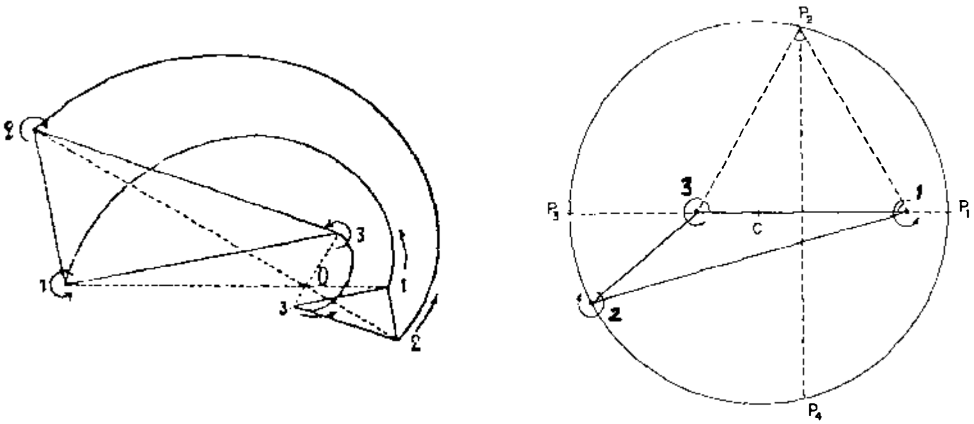

Figure 3 Self-similar motion in which the vortex triangle changes its size but not its shape. In the example on the *left*, taken from Gröbli's dissertation, a self-similar expansion is shown. This figure results for vortices of circulation $m_1 = 3$, $m_2 = -2$, $m_3 = 6$ (only the ratios are important). On the *right* we show a general construction of all configurations that lead to self-similar motion (when the vortex circulations have the appropriate ratios). Vortices *1* and *3* are assumed of the same sign and are initially located as indicated. The circle is centered on the center of vorticity of vortices *1* and *3* (point *C*) and is such that triangle $1P_23$ is equilateral. There are four configurations that move without changing either shape or size. These arise if vortex *2* is started at any of the four positions P_1, P_2, P_3 and P_4. For any other starting point of vortex *2* along the circle a self-similar expansion or collapse of the vortex triangle occurs.

EPILOGUE

Gröbli got his Doctorate in Göttingen. The oral examination was attended by Schwarz, who had moved to Göttingen while Gröbli was in Berlin, and by Johann Benedict Listing (1808–1882), professor of mathematical physics and optics. A written discussion and recommendation by Schwarz exists. The dissertation was printed in *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich.* Gröbli returned to Zürich and worked for six years as an assistant to Frobenius, who had succeeded Schwarz. From 1877–1894 he held the title of *Privatdozent* at the ETH with an official teaching assignment in hydrodynamics. In 1883 he was elected professor of mathematics at the Gymnasium in Zürich.

We are only aware of one other work by Gröbli in fluid mechanics. In 1876 he was awarded a prize for responding to the following University of Berlin prize question: "In all cases where the theory of fluid jets has been carried through, the rigid walls on which the jets appear are plane. Investigate and treat cases in which the walls are curved." The actual work is not available.

Did Gröbli's dissertation have an impact on contemporary science? The answer is clearly affirmative. Kirchhoff mentions it in a footnote in his widely used lectures on mathematical physics, and Lamb cites it in his classic text *Hydrodynamics*. In 1898 D. N. Goryachev, a student of Zhukovsky, published a dissertation in which he treated integrable cases of four-vortex motion—essentially the cases treated almost a century later by Eckhardt (1988) and Rott (1990)! The dissertation (Goryachev 1898) is very similar in style to Gröbli's, which is cited early on.

Later, however, Gröbli's dissertation was largely forgotten until a 1949 paper by the Irish mathematician John Lighton Synge addressed what was, in essence, the converse problem. Given that the three-vortex problem can always be integrated, Synge (1949) inquired about the fundamentally different types of motion that the three interacting vortex points can execute as one varies the vortex strengths.

Although Helmholtz and his contemporaries primarily had classical fluids in mind when they worked out theories and models of vortex motion, the concept of a line vortex has received a range of new applications with the discoveries of superfluidity and superconductivity. Modern interest in the dynamics of a few interacting point vortices, in particular the onset of chaos in the four-vortex problem, was to a large extent stimulated by Novikov's (1975) paper. But these are rather different stories for another time.

Why is the published oeuvre of Gröbli restricted to his dissertation? We can, of course, only speculate. In an obituary he is described as being of very modest character, who would present his scientific work in lectures without publishing it. Could it be that his contact with eminent scientists led him to apply too high a standard to his own work? Another reason, no doubt, was his strong passion for the mountains, where he spent most of his spare time. Since his first tour in 1877, climbing dominated his life. The lecture "Professor Dr. Walter Gröbli as mountain climber" given in 1904 by Dr. A. Lüning to the Swiss Alpine Club clearly shows this. Sixteen years after Edward Whymper, Gröbli climbed the Matterhorn. Sometimes, he would hike up to 70 km per day. Even today his name is mentioned in guidebooks of the Swiss Alpine Club as having opened the first route on the Piz Ela (3339 m) and the first route through the north face of the Tödi (3620 m). He published several descriptions of regions and mountains that he visited.

One June 26, 1903, Gröbli and a colleague were conducting a tour with 16 students, a class from the Gymnasium, to the Piz Blas. They met more snow than expected and, due to the warm day, had to change their plans. While crossing a south-facing slope they were caught in an avalanche. Gröbli and three of his students were thrown over a cliff and killed. The accident was described in *Zürcher Wochenchronik* No. 27 (July 4, 1903).

Gröbli's story intrigues us because of its unique personal aspects, the reputation and stature of his teachers and advisors, and the depth and significance of his dissertation. Personal matters aside, few doctoral students today have the benefit of such stellar faculty, and, truth be known, few will write a thesis that will be the subject of attention a century later.

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