(*Left* to *right*) M. D. Millionshchikov, A. N. Kolmogorov, A. M. Yaglom, and R. Kraichnan at meeting at the Institut de Mécanique Statistique de la Turbulence, Marseille, 1961. (Photo courtesy J. L. Lumley.)

Annu. Rev. Fluid Mech. 1994. 26 : 1–22

A. N. KOLMOGOROV AS A FLUID MECHANICIAN AND FOUNDER OF A SCHOOL IN TURBULENCE RESEARCH

A. M. Yaglom

Institute of Atmospheric Physics, USSR Academy of Sciences, Moscow, Russia

Andrei Nikolaevich Kolmogorov, an extraordinary scientist and quite remarkable man, was born on April 25, 1903, and died on October 20, 1987. According to the Russian usage he was called Andrei Nikolaevich by his colleagues and students (except for a few close friends and relatives who called him simply Andrei); for short he will usually be called AN in what follows. Several expositions of AN's work and life have already been published; see e.g. the long comprehensive paper by Shiryayev (1989), the excellent collaborative obituary edited by Kendall (1990), three volumes of AN's selected works (in accordance with the author's recommendations) supplemented by short comments by AN himself and more detailed commentaries by some of his students and colleagues (Kolmogorov 1985, 1986, 1987; these three Russian books have now been published in English, see Tikomirov 1991, Shiryayev 1992, and Sossinsky 1992), and the Royal Society Volume compiled and edited by Hunt, Phillips, and Williams (1991) on the occasion of the 50th anniversary of Kolmogorov's papers on turbulence (1941a,c). This latter volume is devoted mostly to the modern development of the ideas presented in these papers. Nevertheless, many facts related to AN's scientific and social activities are not known to the majority of the fluid dynamics community; therefore it seems appropriate to publish a paper about him in this *Annual Reviews* volume.

AN is most famous as one of the greatest mathematicians of the 20th century: of the same caliber as such mathematical geniuses as Henry Poincaré, David Hilbert, Hermann Weyl, and John (János, Johann) von

0066-4189/94/0115-0001$05.00

Neumann. His mathematical works are amazing in their profundity and originality combined with unique diversity and breadth; they concern quite different topics which, together with a number of very abstract and sophisticated matters, include many quite practical and easily applicable ideas and results. His brilliant works in mechanics are also famous. AN's theory of locally isotropic turbulence fully transformed the most complicated and practically important part of modern fluid mechanics. His profound studies of ergodic and dynamical system theories gave birth to the splendid KAM-theory (after Kolmogorov, Arnold, and Moser) in the classical mechanics of systems with finite numbers of degrees of freedom—one of the greatest achievements in mathematical natural sciences of this century [see e.g. Arnold's commentary in Kolmogorov (1985) and Moffatt's section in Kendall (1990)]. It is therefore only natural that Abraham & Marsden (1978) include in their well-known book on mechanics AN's photo in a small portrait gallery of the greatest mechanicians of humanity (beginning with Archimedes' portrait). Moreover, many readers of this volume probably do not know that Kolmogorov's scientific interests were not limited to only mathematics and mechanics. His first scientific work was devoted to medieval (15th century) Russian history; this work has just recently been published (Kolmogorov 1992) but is considered very important by leading experts on ancient Russia (see Yanin 1988). Prosody was another part of humanities that attracted Kolmogorov's attention for many years. In Russian magazines and collections of papers on philology and linguistics he published (in part jointly with his students) about a dozen papers devoted to the profound study of style, metrical structure, and rhythm of poetry by various Russian poets (both classical and modern); these papers will be collected in the fourth volume of AN's selected works, which is being prepared now by Nauka Press.

Let us now go on to AN's works on turbulence which are of most interest to the readers of this volume. In his short comments on these works Kolmogorov (1985) wrote:

> I took an interest in the study of turbulent flows of liquids and gases in the late thirties. It was clear to me from the very beginning that the main mathematical instrument in this study must be the theory of random functions of several variables (random fields) which had only then originated. Moreover, it soon also became clear to me that there was no chance to develop a closed purely mathematical theory. For lack of such a theory it was necessary to use some hypotheses based on the results of treatment of experimental data. Therefore it was important to find talented collaborators able to work in such a mixed regime who could combine theoretical studies with the analysis of experimental results. In this respect I was quite successful.

(AN goes on to mention his students A. M. Obukhov, M. D. Millionshchikov, A. S. Monin, and the present author.)

AN's comments require some additional explanation. It is characteristic of AN that, if possible, he always tried to supplement his purely mathematical works with some practical applications. In particular, his profound general theory of Markov processes was supplemented by the solution of some specific problems related to Brownian motion in physical systems (see e.g. Kolmogorov 1986). His study of fundamental theoretical problems of mathematical statistics led him (in the war years) to investigate applications of statistical methods to the theory of ballistics. Before and after the war he sought solutions for statistical problems of quality control, weather forecasting, and so on. In his famous book on foundations of probability theory (Kolmogorov 1933, 1956a) AN was the first to give a rigorous definition of a random function $X(t)$, where t passes through infinitely many values, as a probability measure in the infinite-dimensional space of real functions $x(t)$ specified by the infinite family of multidimensional probability distributions. Two years later Kolmogorov (1935) also provided another method to specify a random function by its characteristic functional. This was an infinite-dimensional generalization of the characteristic functions widely used in probability theory. These works formed the basis for the mathematical theory of random (or stochastic) functions which later became an extensive and highly developed part of modern probability theory.

It was typical of AN that his development of the mathematical theory of random functions aroused his interest in possible applications of such functions. For the case of a scalar (one-dimensional) variable t, some applications are obvious; they are provided by Brownian motion and irregular oscillations of physical systems (here t must be interpreted as time). However, for the case where t is multidimensional, the most natural applications are those related to the study of fields of fluid dynamic variables in turbulent flows. This was the primary reason for AN's interest in turbulence.

AN easily made contacts with people, and being interested in turbulence he participated in many discussions with Soviet experts in fluid mechanics such as L. G. Loitsyanskiy, I. A. Kibel [a former student and collaborator of A. A. Friedmann whose joint paper with Keller (1924) was highly valued by Kolmogorov], and many others. The study of Taylor's paper of 1935 on isotropic turbulence was especially inspiring for AN and led him to ideas about further study of such turbulence. AN felt that such a study had to involve a direct analysis of experimental data and he started searching for a student appropriate for such a task. He consulted a number of fluid mechanics professors and some of them recommended a young mechanical engineer named M. D. Millionshchikov. Millionshchikov had graduated from an oil industry engineering college in his native city of Grozny in the Caucasus (which has recently become famous for turbulent political activities) and in the mid-1930s taught at the Moscow college preparing aviation industry engineers. AN spoke with him and then agreed to be his scientific adviser during his graduate study at the Aviation Industry College. Thus, AN found his first graduate student specializing in the mechanics of turbulence.

AN's second graduate student of the same speciality was A. M. Obukhov. The latter was born in Saratov (a city on the Volga river) on May 5, 1918—exactly one hundred years after the birth of Karl Marx—and in 1935 he entered Saratov University to study mathematics. In 1937 a special competition dedicated to the 20th anniversary of the 1917 October Revolution was announced in Moscow for young scientists of the USSR. AN was one of the judges at the mathematics section. According to his suggestion the first prize in mathematics was given to the unknown young student Obukhov from Saratov University. The prize-winning paper by Obukhov was, in fact, extraordinary: It dealt with multivariate statistical analysis and proposed a new statistical technique which later became known as canonical correlation analysis. The technique is now presented in dozens of textbooks, manuals, and monographs and has many diverse applications; see e.g. the survey by Yaglom (1990) published in the journal issue devoted to the memory of Obukhov. AN naturally took a keen interest in the young student. He arranged Obukhov's transfer from Saratov to Moscow University, took care of him, and after his graduation recommended him for graduate study under his guidance. At that time AN's main interest focused on the mechanics of turbulence and he suggested to Obukhov a theme related to this field. This suggestion happened to be extremely successful. In fact, between high school and university Obukhov spent a year working at the Saratov Meteorological Observatory. (He was not admitted to the entrance exams at Saratov University in 1934 because he was too young: Russian provincial universities are often quite conservative.) Later Obukhov used the results of his meteorological observations as the basis for his first scientific work and henceforth meteorology became his permanent love. (The work on mathematical statistics mentioned above was inspired by Obukhov's reflections about possible ways of treating wind observations.) Since atmospheric air flows are always turbulent, the study of turbulence attracted Obukhov very much and he worked with great enthusiasm and energy. Simultaneously with the beginning of his graduate study Obukhov started working at the Institute for Theoretical Geophysics of the USSR Academy of Sciences which had just been organized by the famous Soviet mathematician (an expert in abstract algebra), polar explorer, theoretical astronomer (author of a new theory for the origin of planetary systems), and member of the USSR Academy of Sciences—Professor Otto Schmidt. This circumstance later proved to be of some importance for the development of turbulence studies in the USSR.

During the war years AN found two more graduate students interested in turbulence: A. S. Monin and the present author. Monin graduated from the Moscow University Mathematics Department in the spring of 1942 and was recommended for graduate study; AN agreed to be his scientific adviser. At that time Monin knew nothing about fluid mechanics and asked for a subject on pure mathematics. However, he was inducted into the army shortly after his graduation and had no chance to start his graduate work. In the army he was admitted to the military meteorological school and ended the war as an officer-meteorologist serving at military airfields. After the war he got in contact with AN who helped to arrange his transfer to the Central Weather Forecasting Institute of the Ministry of Defense in Moscow. There he continued his graduate study under the guidance of AN who naturally gave him a subject related to the study of atmospheric turbulence.

I studied at Moscow University simultaneously with A. S. Monin in both the Physics and Mathematics Departments. As the war came to the USSR in 1941, I tried to join the army but was rejected because of nearsightedness (probably great luck, since most of my friends admitted to the army were killed in the war). In the autumn of 1941 I went to the city of Sverdlovsk together with my parents and continued studying at Sverdlovsk University. AN spent the early war years partially in Moscow and partially in Kazan where the USSR Academy of Sciences was transferred. But he also made several visits to Sverdlovsk which was a big scientific center. During his first visit to Sverdlovsk in the winter of 1941–1942, AN delivered a lecture on the small-scale structure of turbulence at Sverdlovsk University. He already knew me a little since in the spring of 1938 he handed me a prize as one of the winners of the Moscow Mathematical Olympiad for high school students which he helped arrange. (AN was always very active in high school mathematical education and he never forgot the young students he met.) After the lecture he noticed me and invited me to visit him the next day at his hotel. When we met, AN told me that many talented young mathematicians had already perished in the war; therefore he wanted to be informed about any good students who had survived and would be happy to help them. (Later he helped me in arranging my admittance, after my graduation from the university, to the Main Geophysical Observatory—a famous meteorological research institution evacuated from Leningrad to Sverdlovsk during the war.) Then AN turned to a discussion of his lecture at the University and, in fact, gave me another impromptu lecture on turbulence. At that time I was very poorly educated in fluid mechanics and did not understand most of the lecture, though I felt that the topic was very interesting. This situation was not exceptional at all; AN often overestimated the level of his listeners and for them it was often difficult to understand everything he said. It was noted by somebody in Moscow that AN apparently assumed that all the world around him was inhabited by Kolmogorovs! However, for good students he was an excellent lecturer and supervisor since his lectures and speeches were always very interesting and challenging. At the end of our meeting he recommended that I read some papers on stochastic processes (also overestimating my abilities to some extent) and asked me to contact him during his subsequent visits to Sverdlovsk.

After his first visit I met AN again in Sverdlovsk two or three times. In the spring of 1943, when Moscovites who had left the city were invited back by some of the Moscow institutions, AN proposed that I go to Moscow and start graduate study under his guidance. I agreed joyfully and he sent me an invitation from the Steklov Mathematical Institute of the USSR Academy of Sciences. When all the necessary formalities had been overcome I returned to Moscow to become AN's graduate student. During our conversations in Sverdlovsk AN suggested that I study turbulence during my graduate studentship, but in Moscow he changed his mind and proposed a subject related to the theory of Brownian motion. The problem suggested to me was sketched by AN several years earlier (in Kolmogorov 1937) and apparently he selected it because he knew of my interest in statistical physics. (This, in fact, appeared to be important for finding the solution of the problem.) However, my discussions with AN in Sverdlovsk and the job at the Main Geophysical Observatory made me take an interest in turbulence. Therefore during my graduate studentship I participated in a seminar on turbulence headed by AN, and after my thesis defense I agreed to take a job related to research in this field (discussed below).

In 1951 AN acquired one more graduate student specializing in turbulence research: G. I. (Grisha) Barenblatt. He had graduated from the Fluid Dynamics Department of Moscow University and been recommended for graduate study in that department. Barenblatt asked AN to be his scientific adviser.

Let us now go on to the fluid mechanics work by AN and his students. The first short paper by such a student was written by Millionshchikov (1939). It was devoted to the study of asymptotic behavior, as $t \to \infty$, of the solution of an approximate equation for the longitudinal correlation function of isotropic turbulence $B_{LL}(r, t) = \langle u_L(\mathbf{x}+\mathbf{r}, t)u_L(\mathbf{x}, t)\rangle$ (where angular brackets symbolize averaging, u_L is the velocity component parallel to $\mathbf{r}$, and $r = |\mathbf{r}|$), obtained when the third moments of velocity fluctuations are neglected in comparison with the second moments. In the introduction to his paper Millionshchikov formulated for the first time a basic idea of AN: that the fields of fluid mechanical parameters in incompressible turbulent flows must be considered as random fields in a three-dimensional space of points $\mathbf{x}(t)$ and the velocity fields are specified by probability measures in the functional space of all solenoidal vector fields. Millionshchikov's paper of 1939 was mainly a continuation (to some extent a completion) of the results by von Kármán & Howarth (1938), related to the same equation. More original was the following paper by Millionshchikov (1941) where the exact form of the equations for the velocity correlations of isotropic turbulence was used. These equations were supplemented in this paper by equations for the third-order moments of velocity fluctuations, where the fourth-order moments of these fluctuations—entering the equations for third-order moments—were approximately expressed through correlation functions with the aid of the assumption that the random velocity field has a Gaussian (or normal) probability distribution. Such a procedure leads to a closed system of equations for second- and third-order correlation functions. It was the first two-point closure of dynamic equations for turbulent flows. The corresponding approximation is now usually called the *quasi-normal* or *zero-fourth-cumulant* approximation and the hypothesis leading to this approximation is called *Millionshchikov's* or the *zero-fourth-cumulant hypothesis*. When I first read Millionshchikov's paper, I was sure that the idea of applying such a hypothesis to the study of turbulence was due to Kolmogorov. In fact, Millionshchikov was not an expert in the theory of random functions. However, in his paper of 1935 AN gave the first rigorous specification of Gaussian random functions and emphasized that all the moments of such a function can be expressed through their first- and second-order moments. Later my late friend Professor S. V. Fomin—who was AN's graduate student simultaneously with Millionshchikov and was present at several conversations between AN and the latter—confirmed my guess about the originator of the zero-fourth-cumulant hypothesis.

AN himself occasionally used the term "Millionshchikov's hypothesis"; maybe it was even first proposed by him. This does not contradict the statement above since AN often attributed his results to some of his students or collaborators and was overly delicate in mentioning the contribution of others. As an example, AN's paper on information theory (Kolmogorov 1956b) gives an equation for ε-entropy of some signal which, according to the author, "was found by Yaglom." However, I first learned of this equation after reading this paper. I asked AN about this reference to me in his paper of 1956 and he answered with a vague remark that the given equation follows from some remark of mine made sometime in a discussion with him. However, I do not remember any such remark and think that he was mistaken. Moreover, in 1947 at a meeting of the special seminar on the theory of branching stochastic processes led by AN (the notion of "branching processes" first appeared at this seminar), AN showed me the manuscript of the fundamental paper on the subject (Kolmogorov & Dmitriev 1947). To my surprise he asked me to excuse him for not including my name in the list of authors saying: "Of course, you and E. B. Dynkin (another student of his) also contributed to this work by your remarks during the discussion, but the remark by Kolya Dmitriev was more important and he is younger than both of you and needs to be encouraged; therefore I included only his name in the list of authors." I do not remember the contribution of Dynkin, but I knew well that my remark was quite minor and of no real importance. As to the contribution by Dmitriev [a talented young student who shortly after 1947 changed his speciality to applied physics, played a rather important part in the development of the Soviet hydrogen bomb, and was in this respect mentioned in Sakharov's memoirs and wrote his own memoirs on Sakharov; see Lebedev Physics Institute (1991), pp. 167–70], though clearly more important than mine, it seemed to me that it was also insufficient for coauthorship. Some other examples of AN's overestimation of the works by his colleagues, which are very typical of him, can be found in Kendall (1990, p. 34).

AN's first papers on turbulence mechanics appeared in 1941. Two remarkable short notes (Kolmogorov 1941a,d) undoubtedly marked the greatest achievement in the theory of turbulence since the time of O. Reynolds. They cardinally changed this theory and are basic to all subsequent developments in the field during the second half of this century. The results of these notes are now well known since they are presented in detail and discussed in many dozens of monographs and surveys; see e.g. the first survey and the first monograph in this field by Batchelor (1947, 1953), the long handbook by Monin & Yaglom (1975), the survey prepared by Yaglom (1981) on the occasion of the 40th anniversary of Kolmogorov's notes, and a special volume compiled by Hunt, Phillips & Williams (1991) celebrating the 50th anniversary of these same works. It is not necessary here to consider the content of these papers in any detail. However, the relation between these works and the paper of Obukhov (1941b) requires some comments.

Before the completion of the investigation described in his papers of 1941 AN presented at a seminar some preliminary arguments about the self-similarity of the cascade process of breakdown of turbulent eddies. He stated that self-similarity must imply proportionality of the longitudinal structure function $D_{LL}(r) = \langle[u_L(\mathbf{x}+\mathbf{r})-u_L(\mathbf{x})]^2\rangle$, $r = |\mathbf{r}|$, to some power r^m of the distance r for a wide range of r values. However, at that time AN did not know how to determine the exponent m. These preliminary considerations were known to Obukhov but AN and his graduate student were at that time separated and continued working quite independently. This independent work led both men to the determination of the value of m, but each used a different method. AN formulated (in Kolmogorov 1941a) two similarity hypotheses describing the universal equilibrium regime of small-scale components in any turbulent flow at high enough Reynolds number. He justified these hypotheses by clearly formulated heuristic physical arguments, related to general ideas about the mechanism of developed turbulence stated during the 1920s by L. F. Richardson. The arguments seemed to be convincing but they contained no proof (there are no mathematical proofs at all in his paper) and required experimental verification. The Kolmogorov hypotheses severely restrict the form of all the statistical characteristics of small-scale turbulence and hence have many special corollaries. In particular, as applied to $D_{LL}(r)$ (this particular application was considered by AN in the 1941a paper) the hypotheses, supplemented by simple dimensional arguments, imply that $m = 2/3$ and also give a crude estimation of the range of r values where $D_{LL}(r) \sim r^{2/3}$. In another paper on the same subject (Kolmogorov 1941d) AN supplemented the results about $D_{LL}(r)$ given in his first paper by corollaries from the dynamic equations and also tried to compare the results obtained with the experimental data available in 1941 (which, as it became clear later, were not fully appropriate for this aim). On the other hand Obukhov (1941b) based his derivation on a model equation (which was established with the aid of a special semiempirical hypothesis) for the spectrum of turbulence $E(k)$. [The spectrum of disordered fluctuations was first introduced in the probabilistic context by Kolmogorov (1940) for the case of fluctuations represented by a random function of one variable; the spectrum of turbulence $E(k)$ related to the random velocity field $\mathbf{u}(\mathbf{x})$ first appeared in the paper by Obukhov (1941a).] Obukhov's semiempirical hypothesis was, of course, nonrigorous and Obukhov showed an amazing physical intuition by extracting from his model equation one consequence which is, in fact, universal and is independent of the particular form of the closure hypothesis used. Obukhov's result states that $E(k) \sim k^{-5/3}$ over a wide range of wave numbers k. It is easy to show that this is equivalent to the relation $D_{LL}(r) \sim r^{2/3}$. (This follows from the relation between $E(k)$ and $D_{LL}(r)$ which is a version of a Fourier transformation.) However, this is only one particular consequence of Kolmogorov's general theory, whose essence consists of two similarity hypotheses. True, Obukhov (1941b) also indicated at the end of his paper another important result of the Kolmogorov theory—namely the "4/3 power law" for the eddy diffusivity that characterized the relative turbulent diffusion of different length scales. This last law was empirically established much earlier by Richardson (1926), and in his theoretical derivation of it Obukhov referred not to general principles but only to his result as it related to the spectrum $E(k)$. Hence, strictly speaking, the content of the paper by Kolmogorov (1941a) was considerably more general than the (very interesting and important but nevertheless only particular) results given by Obukhov. However, AN here also demonstrated his high respect for the works of others, always stressing that the general theory of small-scale turbulence was a joint creation of his and Obukhov (see e.g. Kolmogorov 1949, 1962a,b).

In 1941 AN published one more paper (Kolmogorov 1941c) devoted to a special consequence of the general theory presented in Kolmogorov (1941a). This paper deals with an idealized model of isotropic turbulence in unbounded three-dimensional space and employs an assumption about the existence of the bounded and time-independent Loitsyanskii integral Λ. The last assumption now seems to be questionable (see e.g. Monin & Yaglom 1975, section 15); therefore the results implied by this assumption must now be considered as unreliable.

In early 1941 AN was absorbed in the study of turbulence, but on June 22 of that year Germany attacked the USSR. The war made working conditions in our country much worse for everybody. Nevertheless, AN continued his work and in January of 1942 he delivered a report on equations of turbulent motion at the first meeting of the Section of Physical and Mathematical Sciences of the USSR Academy of Sciences held in Kazan. (This Section was headed by Kolmogorov from 1939, when he was elected to the Academy, to the end of 1942. The USSR Academy of Sciences was transferred to the city of Kazan on the Volga shore in the Autumn of 1941 when the German armies were very close to Moscow.) The abstract of the report by AN was later published (Kolmogorov 1942) and it was quite remarkable. Here AN did not limit himself to investigation of only small-scale fluctuations, but he tried to construct an approximate (model) system of equations describing the large-scale properties of turbulent motions responsible for most of the turbulence effects that are interesting to engineers. However, in contrast with the early semiempirical theories of turbulence developed in the 1920s and early 1930s by L. Prandtl, T. von Kármán, and G. I. Taylor, AN did not limit himself to Reynolds equations for mean velocity components $\langle u_i \rangle$, $i = 1, 2, 3$, closed by hypothetical algebraic equations for Reynolds stresses. Instead he introduced additional differential equations for the energy of the fluctuating motion b and its typical frequency ω (related to the turbulence length scale L, eddy viscosity K, and energy dissipation rate ε by the equations $L \sim b^{1/2}/\omega$, $K \sim b/\omega$, and $\varepsilon \sim b\omega$). The resulting closed system of equations determined the variables $\langle u_i \rangle$, b, and ω (and hence also L, K, and ε) which are the main characteristics of turbulent motion.

In the pre-computer era it was natural to note, as AN did in his 1942 paper, that the solution of the proposed system of differential equations "presents great difficulties." However, AN stated that these equations were nevertheless solved by him and his collaborators for the case of plane Couette flow and that some preliminary results were also obtained for flow in a circular pipe. In his interesting commentary on AN's paper of 1942, D. B. Spalding (1991) noted that a source term was missing in the equation proposed by AN for ω. Spalding assumed that in the performed numerical solutions this lack was probably compensated by introducing a special boundary condition at the wall; if so, then AN had anticipated the practices of a number of later workers. However, it is also possible that the source term is missing because of a misprint in the text (there are a number of obvious misprints in this poorly printed war-publication of 1942). Spalding also noted the incompleteness and inappropriateness of the emphasis on the strong similarity between the works by Kolmogorov (1942) and Prandtl (1945) in many reviews of model equations of turbulent flows (e.g. in Harsha 1977 where the term "Prandtl–Kolmogorov models" is widely used). In fact, Prandtl developed a one-equation model of turbulence in which the equations for the mean velocity components were supplemented by an equation for the turbulent energy (and therefore the length scale L, or K, or ε, must be determined here by some crude guess), while AN was the first who proposed a two-equation model, which was closed and required no additional hypotheses. Of course, the models that use two additional equations are more complicated and require more difficult computational work than one-equation models but the former give a more accurate description of turbulent characteristics and therefore are the most popular at present. Apparently the first rather crude one-equation model was proposed by Zagustin (1938) [the reference to this early work was included in the book by Monin & Yaglom (1971) at the suggestion of AN]; after Prandtl (1945), one-equation models equivalent to his were independently proposed and used for turbulent computations by several authors during the 1950s and 1960s (see Spalding 1991). Note also that when after the war AN proposed to his graduate students Monin and Barenblatt the particular computational problems of mechanics of turbulence, they both used (at the recommendation of AN) the simplified one-equation version of equations given in Kolmogorov (1942) (see Monin 1950; Barenblatt 1953, 1955; or Monin & Yaglom 1971, Sections 6.6 and 6.7). However, in the late 1960s and 1970s a number of scientists (Harlow and Nakayama, Spalding, Saffman, and some others) working independently from each other and from AN (whose work of 1942 was then unknown in the West) proposed two-equation models quite close to that of AN (see again Spalding 1991). The high popularity of such models clearly shows that in his paper of 1942 wonderfully AN anticipated the future developments needed to satisfy practical workers. This penetrating insight, clearly seen in many of his papers, often places these papers much ahead of their time.

After 1942 much time passed before AN's next publication on turbulence appeared. The war years were, of course, not propitious at all for pure science but, nevertheless, AN continued to preserve his keen interest in the mechanics of turbulence. Around 1944, when it became clear that the war was going to end and Obukhov returned to Moscow from Kazan together with the Institute of Theoretical Geophysics where he worked, AN arranged a small seminar on turbulence which was held first at Moscow University (in the old building in downtown Moscow) and then at the Institute of Theoretical Geophysics. At the beginning the seminar was not regular and alternated with meetings devoted to other problems, but from 1946 onward meetings of the turbulence seminar were held regularly every fortnight. AN delivered a number of reports at the seminars—partially original and partially reviewing publications by other authors. I remember well his report on the spectral theory of isotropic turbulence where the spectral formulation of Millionshchikov's results of 1939 was presented and the spectral expression for the Loitsyanskii integral was given. This report was very helpful to me since it permitted me to simplify considerably the analysis of the final period of decay for isotropic turbulence in a compressible fluid. I started the investigation of this problem in Sverdlovsk during my work at the Main Geophysical Observatory, but without the spectral approach the analysis was very cumbersome and only after AN's report could I complete the work and publish it as my first paper on turbulence (Yaglom 1948). However, AN never published any results from this report of his; only his expression for the Loitsyanskii integral was presented in my paper of 1948 (of course, with a direct reference to AN) and still later it was independently found by Batchelor (1949). I remember also AN's review report on the derivation of skin friction laws by assuming the existence of an overlap of the wall and outer sublayers in turbulent flows. Such a derivation is, in fact, a simple combination of similarity and dimensional arguments; now it is quite popular and is presented in many textbooks (usually with a reference to Millikan 1939). However, AN delivered his report in the mid-1940s, when this approach was unknown to most researchers. Moreover, he based his talk on a rather unpopular paper by von Mises (1941) which contains a very clear and logical presentation of Millikan's results supplemented by some useful original remarks.

Together with AN's students interested in turbulence, many applied scientists and engineers participated in the turbulence seminar. Quite often these participants presented their experimental results and AN was always eager to discuss such reports, to explain the theoretical deduction related to the data, and to take part in comparison of experimental results with theory. Several publications of AN (Kolmogorov 1946a,b, 1949, 1952, 1954) were devoted to discussions of engineering papers, to criticism of errors committed by the authors, and to theoretical explanations of the results obtained. In particular, in Kolmogorov (1946b, 1954) AN criticized the views of the well-known Russian hydrologist M. A. Velikanov on the transport of suspended sediment by turbulent flows in rivers. In this respect he proposed a new theory of such transport which was then developed quantitatively by his student Barenblatt (1953, 1955). In Kolmogorov (1946a, 1952) he explains errors committed by some authors in the "theoretical derivation" of skin friction laws in circular tubes. Of great interest is his 1949 note (the only one of the papers cited above that AN felt should be included in the collection of his selected works published in 1985), which reflects AN's reaction to one purely experimental report at the seminar. It contains an elegant application of dimensional analysis and universal equilibrium theory from Kolmogorov (1941a) to the physical problem of dispersion of drops of a liquid in the turbulent flow of another liquid where the surface-tension stress plays an important part. Many other applications of the same theory were discussed at the seminar; AN participated actively in such discussions but let the others publish the results.

Note now that Obukhov tried to arrange experimental verifications of the laws, found in AN's and his own papers of 1941, immediately after the discovery of these laws; some preliminary results of the very first measurements were published in Obukhov (1942) but then the war intervened. When the war came to an end in 1945 Obukhov returned to his old idea which was also supported by AN and by Professor O. Yu. Schmidt, Director of the Institute of Theoretical Geophysics. It was then decided to arrange a special Laboratory of Atmospheric Turbulence at the Institute of Theoretical Geophysics and Schmidt proposed that AN head the laboratory. In short comments on his works on turbulence included in Kolmogorov (1985) AN gave the following description of this experience:

> In 1946 O. Yu. Schmidt proposed me as the head of the Laboratory of Atmospheric Turbulence at the Institute of Theoretical Geophysics, USSR Academy of Sciences. In 1949 I stopped working there and Obukhov replaced me as the head of the laboratory. I did not participate directly in the experimental work but spent much energy on computational and graphical treatment of the data obtained by other investigators.

Some details must be added to this short description. The Institute of Theoretical Geophysics was originally placed in the former private house of a rich Moscow merchant named Lepekhin, in a part of Moscow that was traditionally inhabited by merchants. The house was not large enough for a big scientific institute and the only room which could be found for the new laboratory was a former merchant's bathroom with all its walls covered with white tiles. Moreover, there was also a grid on the window since before the laboratory was placed there the room had been occupied by an office where classified information had been stored. Two desks and several chairs were the only furniture which could be placed in the small room. The first staff of the laboratory consisted of AN and only three other people: A. M. Obukhov, P. A. Kozulyaev (a former graduate student of AN whose candidate dissertation was devoted to the extrapolation of time series), and myself. This selection was quite accidental. Kozulyaev was jobless at that moment and AN invited him to help with the treatment of the data; one year later he left the laboratory and became a professor in one of the Moscow engineering colleges. I was then more interested in turbulence than Kozulyaev but my inclusion in the laboratory staff was also in some sense accidental since I had at that time quite another plan for my future. During my graduate study I solved the problem proposed by AN in one year. AN told me that this was enough for the candidate dissertation and therefore he could arrange my dissertation defense which would end my graduate study. I answered that I would prefer to preserve the position of a graduate student (having a small stipend and much free time) as long as possible; hence I asked him to delay my defense until the normal end of my studentship. (The normal period for a graduate study in the USSR is three years.) AN approved my decision and said: "I can understand you. I spent five years as an undergraduate student and four years as a graduate student and was happy to have that opportunity. By the end of my graduate study I had 18 published works." So I was permitted to do what I wanted for two years. During these years I participated in AN's seminars on turbulence and stochastic processes but also attended all the meetings of theoretical physics seminars led by L. D. Landau and by I. E. Tamm (both future Nobel laureates). Simultaneously I worked jointly with Professor I. M. Gelfand on the general theory of relativistic wave equations: Theoretical physics was then my main passion. After the end of my graduate studies AN proposed several possible jobs to me (there were few young candidates of sciences in 1946 and many free vacancies), but more attractive to me was the proposition by Tamm to join the Theory Department of the P. N. Lebedev Physics Institute. However, it upset me to learn that I would have to spend some of my time on "applied problems" (i.e. on the atomic bomb project) in this department. I have already written [in my memoirs about Sakharov; see Lebedev Physics Institute (1990), pp. 660–74] that I probably would have reacted negatively to any proposal to participate in such a project under any government, but in the case of the Stalin regime, which I considered as being very oppressive and dangerous to all mankind, such participation was impossible for me. Therefore, I agreed to take the job at the Institute of Theoretical Geophysics which had nothing to do with military problems. At the beginning I considered this job as a temporary one and planned to return to theoretical physics in a year or two when the atomic bomb problem ceased to be so urgent. However, it did not cease to be urgent for a long time and, moreover, political development in the USSR made changing jobs impossible for many years. Therefore, I remained working in the same laboratory (of the same Institute which, however, changed its name twice after 1946) till today. (Maybe it was luck. Who knows?)

Two months after organization of the Turbulence Laboratory three more people joined: two young women working as secretaries and laboratory assistants and a qualified technician from the P. N. Lebedev Physics Institute, USSR Academy of Sciences, who immediately began constructing a hot-wire anemometer under the direction of Obukhov and later was responsible for measurements of wind velocity and temperature fluctuations in the atmosphere. AN usually spent one day a week in the laboratory heading the seminar, discussing current work, and looking through the collected experimental data which always interested him very much. Between AN's visits to the laboratory, Obukhov and I often met him at Moscow University and sometimes visited him either at home or at his country house in the village of Komarovka near Moscow. We told AN about our work and discussed the most urgent problems. Such meetings with AN were during some periods more frequent and at other times less frequent but they continued until his last illness.

The replacement in 1949 of AN by Obukhov as the head of the Turbulence Laboratory at first changed nothing. It was a purely formal act aimed at increasing Obukhov's salary: The day by day laboratory work was directed by Obukhov from the very beginning and the general guidance continued to be AN's duty. However, in the early and mid-1950s topics unrelated to turbulence (superpositions of functions, ergodic theory, Hamiltonian mechanics) became AN's main scientific interests. He naturally began to give less attention to work at the Turbulence Laboratory and stopped attending the laboratory seminar.

It is, however, worth noting that AN's approach to the mechanics of turbulence and his general ideas directed the works of his students (and of the students of his students) even during periods when AN himself was engaged on other research problems. One of the most important achievements of the Turbulence Laboratory was the development by Obukhov (1949) of a universal equilibrium theory for scalar fields in turbulent flows. This theory is again based on intuitive physical arguments and the paper by Obukhov is very close in style to the classical paper by Kolmogorov

(1941a). Obukhov's paper of 1949 immediately stimulated an attempt to also apply the approach presented in Kolmogorov (1941d) (see Yaglom 1949) to scalar fields. The very useful similarity theory of turbulence in thermally-stratified boundary layers developed by Monin & Obukhov (1954) has many applications (see e.g. Monin & Yaglom 1971, chapter 4) and is a generalization of the classical wall laws for nonstratified turbulent boundary layers often used by AN in his reports and discussions. Knowledge of the theory by Monin & Obukhov prompted Kader & Yaglom (1978) to apply a similar approach to the analysis of pressure-gradient turbulent wall flows; AN's old report on the derivation of skin-friction laws inspired the derivation of universal heat- and mass-transfer laws for turbulent wall flows (Kader & Yaglom 1972). The paper by Novikov (1971), a former student of Monin, contains a direct development of the ideas given in Kolmogorov (1962a,b) (these ideas will be discussed below); the dimensional analysis of the dynamics of planetary atmospheres by Obukhov's student Golitsyn (1973) reflects the general approach to turbulence problems often popularized by AN.

Let us now return to works by AN himself. The next peak of AN's activity in turbulence research came in the early 1960s. In 1961 two big International Meetings on the Mechanics of Turbulence were arranged one after another in Marseille by the IUTAM (International Union of Theoretical and Applied Mechanics) and the IUGG (International Union of Geodesy and Geophysics). AN was invited to both meetings together with a group of his former students. AN was always fond of France, especially Southern France. (He was less enthusiastic about Germany; probably this was partially connected with memories of the Second World War but my impression was that apparently some experience from his first visit to Germany and France in the early 1930s also contributed to his attitude toward these two countries.) Therefore he was happy to have the opportunity to visit Marseille again. Scientifically the Marseille meetings were also of great interest to him. In preparation for these meetings he showed again great interest in the works of the Turbulence Laboratory he had founded fifteen years before.

Obukhov prepared for the Marseille meeting a report stimulated by the data of A. S. Gurvich who had been working in our laboratory and was engaged in measurements of the spectra of wind velocity fluctuations. Gorvich's data show that although high frequency spectra measured at the same point in the atmosphere during two closely-spaced time intervals both agree well with the relation $E(k) \sim k^{-5/3}$, the proportionality coefficients are often quite different. According to the theoretical results obtained by AN and Obukhov in 1941 the proportionality coefficient is equal to $C\varepsilon^{2/3}$, where C is a universal constant and ε is the rate of energy dissipation. Therefore the data by Gurvich showed that the dissipation rate ε has a strongly fluctuating time dependence in the Earth's atmosphere. It was clear to Obukhov that fluctuations in ε must affect the structure functions (i.e. the mean values of the squared velocity differences at two points) and the spectra (the Fourier transforms of the structure functions). Referring to Kolmogorov's paper (1941b) on the probability distribution of the sizes of particles under fragmentation, Obukhov assumed that the dissipation rate ε is a random variable having a logarithmically normal probability distribution. Moreover, he also assumed that the structure function $D_{LL}(r)$ depends on $\bar{\varepsilon}_r$—the dissipation rate averaged over a sphere of radius r. Under this assumption he evaluated the influence of the dissipation-rate fluctuations on the value of the velocity structure function $D_{LL}(r)$ over an inertial range of distances r [where $D_{LL}(r) \sim r^{2/3}$ according to the Kolmogorov theory of 1941]. Taking the fluctuations into account, Obukhov obtained a result that differed from the classical "2/3 power law" of the old theory by a correction factor of a special form. Obukhov told AN about this result and the latter was very interested in it and asked for a copy of the manuscript of the prepared report. Obukhov and Kolmogorov then parted with each other and were not to meet again until Marseille: AN could not fly because of an ear problem and therefore he went to France by train separately from the other Russian participants at the meetings. Arriving in Marseille AN showed Obukhov his own manuscript which contained an important development of Obukhov's ideas.

AN's paper (Kolmogorov 1962a,b) presented as a report at the Colloquium on Turbulence in Marseille is close in some respects to his earlier 1941 paper (Kolmogorov 1941a). The new paper is also very physical in its style; it contains no mathematical proofs or analytical derivations but instead suggests general similarity hypotheses justified by heuristic arguments. The first two new hypotheses stated more precisely the two similarity hypotheses from Kolmogorov (1941a); they refer to probability distributions not for velocity differences but for the ratios of such differences related to two pairs of points. (By the way, the mathematical theory necessary for the rigorous presentation of Kolmogorov's new theory sketched in AN's papers of 1962 has not yet been developed. This theory must deal with random fields that have probability distributions for the ratios of two differences of field values at different points that are invariant under all spatial similarity transformations.) These two hypotheses were supplemented by a third more special hypothesis postulating the logarithmic normality of the dissipation rate ε and indicating the form of the variance of $\log \bar{\varepsilon}_r$. The three hypotheses imply corrected relations for spectra, structure functions of various orders, and many other statistical characteristics of turbulence. The corrected relations include, in particular, Obukhov's result for the second-order structure function (which was also presented in Marseille and later published; see Obukhov 1962a,b) and differ from the conclusions of AN's theory of 1941 by additional factors depending on the Reynolds number. Unlike AN's theory of 1941 the new theory proposed by AN and Obukhov twenty years later does not lead to any entirely new results but implies only some minor corrections to the known relations which have been, up until now, unimportant for real applications. However, this new theory was a very significant step toward understanding the physical mechanism of developed turbulence and therefore was crucial to most of the further developments in turbulence theory (see e.g. Hunt et al 1991 and Meneveau & Sreenivasan 1991).

Both Marseille meetings were a great success. They collected many brilliant scientists of all generations from many countries of the world including such giants as T. von Kármán, G. I. Taylor, and A. N. Kolmogorov. The American delegation included representatives from at least four generations (the great von Kármán, his student H. W. Liepmann, Liepmann's student S. Corrsin, and Corrsin's student J. L. Lumley). The USSR delegation included Kolmogorov (this was the first and only time when AN actively participated in an international meeting on turbulence), his two pre-war students M. D. Millionshchikov and A. M. Obukhov, and me—a war-years student. Such a composition had a flavor of Khrushchev's liberalization (for me it was the first time I was permitted to attend a meeting in a "capitalist country"), but nevertheless the authorities did not permit Monin to go with us—apparently to show that "high authorities" continue to exist. There were many very interesting reports at the meetings but my feeling was that a short report by AN and an accompanying paper by Obukhov (presented at the second geophysical meeting) were considered by many participants to be the most exciting and sensational. After these talks G. K. Batchelor said that they were worthy of publication not only in the Proceedings of the meetings but also in the Journal of Fluid Mechanics; due to his insistence both papers were published in 1962 twice (1962a are Proceedings publications, 1962b—publications in JFM).

His participation in many scientific discussions at the Marseille meetings and the great success of his report there again aroused AN's interest in the mechanics of turbulence. After returning home from Marseille AN organized the new USSR Seminar on Turbulence within the framework of the Laboratory of Statistical Methods, Moscow University, which he established in 1960 and then headed till his death. The seminars took place in a big hall of the new Moscow University Laboratory Building at Lenin (Vorobjev) Hills in Moscow. Often more than a hundred people participated in these meetings. Both original and review reports were presented at the Seminar and their influence on the development of the mechanics of turbulence in our country was quite substantial. AN led the discussions and always actively participated in them. Some of his deductions based on discussions at the seminar were formulated by AN in his survey lecture on "Experimental and theoretical methods in the study of turbulence," read in May 1965 at the meeting of the Moscow Mathematical Society (AN was then President of this Society). The last report presented by AN at the meetings of this Society was also devoted to turbulence; it was read in January 1978 and was entitled "Remarks on the statistical solutions of Navier-Stokes equations" (see Kolmogorov 1978).

At one of the specialized meetings at the Laboratory of Statistical Methods in the mid-1960s I presented a short review of an old paper by E. Hopf (1952) on the equation for the characteristic functional of the fluctuating velocity field. (This paper was for some reason poorly known in the USSR.) AN then told me that he understood long ago that the Navier-Stokes equations must imply a linear functional differential equation for the characteristic functional of the velocity field. (Let us recall that the concept of a characteristic functional was introduced by AN in his paper of 1935.) "However," continued AN, "I did not try to determine the exact form of this equation since I could not see what applications such an equation could have." AN also stressed that without any theorems about conditions guaranteeing the existence and uniqueness of the solution of an initial value problem for functional differential equations, such equations continue to be useless. Later AN tried to interest M. I. Vishik—a known expert on both functional analysis and partial differential equations—in mathematical problems related to Hopf's equation. Vishik was very interested in these problems, he enlisted some students of his, and began a profound study of the mathematical problems of turbulence. During this study he regularly consulted AN and their long conversations played a very important part in the development of the work. The results of these investigations are summarized in the monograph by Vishik & Fursikov (1988)—which never would have appeared without AN's inspiration.

Renewal of AN's interest in turbulence after 1961 also prompted him to accept Monin's proposal to take part in two voyages (in 1970 and 1971–1972) of the big scientific research ship "Dmitrii Mendeleyev" which set out to study oceanic turbulence. AN had the position of a scientific supervisor on these voyages (together with A. S. Monin). He was mostly engaged during the voyages in the development and applications of improved methods of spectral analysis. These methods permitted one to minimize interference due to the noise at neighboring frequencies, and in principle could be applied even to the spectral analysis of nonstationary (slowly evolving) processes. Another contribution of AN to the joint experimental work consisted in estimating the necessary averaging times, duration of realizations, discretization intervals, etc. He also evaluated the reliability and statistical representativeness of experimental values obtained. AN very much enjoyed his participation in the oceanic voyages and always remembered them with great pleasure and some pride. After the first voyage he presented the scientific impressions which it produced in a survey report at the Moscow Mathematical Society entitled "Statistical fluid mechanics of the ocean."

In the late 1970s the scientific interests of AN again shifted to subjects unrelated to fluid mechanics. Mathematical logic and the foundations of probability and information theories, mathematical linguistics, and the problems of high school mathematical education took most of AN's time near the end of his life. Moreover, in old age he suffered from Parkinson's disease and from an eye illness that made him almost blind. Nevertheless, he tried to work practically until the end. The death of this great scientist on October 20, 1987 was undoubtedly a great loss to all mankind. For the fluid mechanics community his death marked the end of a whole epoch, for AN's essential contributions changed the very content and style of our science.

Literature Cited

Abraham, R., Marsden J. E. 1978. *Foundations of Mechanics*. Reading: Benjamin. 2nd ed.

Barenblatt G. I. 1953. Motion of suspended particles in a turbulent flow. *Prikl. Mat. Mekh.* 17: 261–74

Barenblatt, G. I. 1955. Motion of suspended particles in a turbulent flow occupying a half-space or plane channel of finite depth. *Prikl. Mat. Mekh.* 19: 61–68

Batchelor, G. K. 1947. Kolmogorov's theory of locally isotropic turbulence. *Proc. Cambridge Philos. Soc.* 43: 533–59

Batchelor, G. K. 1949. The role of big eddies in homogeneous turbulence. *Proc. R. Soc. London Ser. A* 195: 513–32

Batchelor, G. K. 1953. *The Theory of Homogeneous Turbulence*. Cambrige: Cambrige Univ. Press

Golitsyn, G. S. 1973. *An Introduction to Dynamics of Planetary Atmospheres*. Leningrad: Gidrometeoizdat. 104 pp.

Harsha P. T. 1977. Models of kinetic energy transfer. In *Handbook of Turbulence*, *Vol. 1: Fundamentals and Applications*, ed. W. Frost, T. H. Moulden, Chap. 8. New York: Plenum

Hopf E. 1952. Statistical hydromechanics and functional calculus. *J. Rat. Mech. Anal.* 1: 87–123

Hunt, J. C. R., Phillips, O. M., Williams, D., eds. 1991. *Turbulence and Stochastic Processes: Kolmogorov's Ideas 50 Years On*. London: Royal Society. 240 pp.

Kader, B. A., Yaglom, A. M. 1972. Heat and mass transfer laws for fully turbulent wall flows. *Int. J. Heat Mass Transfer* 15: 2329–51

Kader, B. A., Yaglom, A. M. 1978. Similarity treatment of moving-equilibrium turbulent boundary layers in adverse pressure gradients. *J. Fluid Mech.* 89: 305–42

Keller, L. V., Friedmann, A. A. 1924. Differentialgleichungen für die turbulente Bewegung einer kompressiblen Flüssigkeit. *Proc. 1st Int. Congr. Appl. Mech., Delft*, pp. 395–405

Kendall, D. G., ed. 1990. Obituary: Andrei Nikolaevich Kolmogorov (1903–1987). *Bull. London Math. Soc.* 22: 31–100

Kolmogorov, A. N. 1933. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Berlin: Springer-Verlag

Kolmogorov, A. N. 1935. La transformation de Laplace dans les espaces linéaires. *C. R. Acad. Sci. Paris* 200: 1717–18

Kolmogorov, A. N. 1937. Zur Umkehrbarkeit der statistischen Naturgesetze. *Math. Ann.* 113: 766–72

Kolmogorov, A. N. 1940. Curves in a Hilbert space that are invariant under the one-parameter group of motions. *Dokl. Akad. Nauk SSSR* 26: 6–9

Kolmogorov, A. N. 1941a. Local structure of turbulence in an incompressible fluid at very high Reynolds numbers. *Dokl. Akad. Nauk SSSR* 30: 299–303

Kolmogorov, A. N. 1941b. Logarithmically normal distribution of the size of particles under fragmentation. *Dokl. Akad. Nauk SSSR* 31: 99–101

Kolmogorov, A. N. 1941c. Decay of isotropic turbulence in an incompressible viscous fluid. *Dokl. Akad. Nauk SSSR* 31: 538–41

Kolmogorov, A. N. 1941d. Energy dissipation in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* 32: 19–21

Kolmogorov, A. N. 1942. Equations of turbulent motion of an incompressible fluid. *Izv. Akad. Nauk SSSR Ser. Fiz.* 6: 56–58

Kolmogorov, A. N. 1946a. On the resistance law for turbulent flows in smooth tubes. *Dokl. Akad. Nauk SSSR* 52: 669–71

Kolmogorov, A. N. 1946b. Comments on the paper by A. M. Velikanov "Transfer of suspended alluvium by a turbulent flow." *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk* No. 5: 781–84

Kolmogorov, A. N. 1949. On the fragmentation of drops in a turbulent flow. *Dokl. Akad. Nauk SSSR* 66: 825–28

Kolmogorov, A. N. 1952. On the resistance law and velocity profile for turbulent flows in tubes. *Dokl. Akad. Nauk SSSR* 84: 29–30

Kolmogorov, A. N. 1954. New variant of the gravitational theory of motion of suspended alluviums. *Vestn. Mosk. Gosud. Univ.* No. 3: 41–45

Kolmogorov, A. N. 1956a. *Foundations of the Theory of Probability*. New York: Chelsea. 2nd English ed.

Kolmogorov, A. N. 1956b. On the Shannon theory of information transmission in the case of continuous signals. *IEEE Trans. Inform. Theory* IT-2: 102–8

Kolmogorov, A. N. 1962a. Précisions sur la structure locale de la turbulence dans un fluide visqueux aux nombres de Reynolds élevés (in French and Russian). In *Mécanique de la Turbulence* (Coll. Int. du CNRS à Marseille), pp. 447–58. Paris: CNRS

Kolmogorov, A. N. 1962b. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds numbers. *J. Fluid Mech.* 13: 82–85

Kolmogorov, A. N. 1978. Remarks about statistical solutions of Navier—Stokes equations. *Usp. Mat. Nauk* 33(3): 124 (Abstr.)

Kolmogorov, A. N. 1985. *Mathematics and Mechanics*. Moscow: Nauka. 469 pp. (In Russian)

Kolmogorov, A. N. 1986. *Probability Theory and Mathematical Statistics*. Moscow: Nauka. 534 pp. (In Russian)

Kolmogorov, A. N. 1987. *Information Theory and the Theory of Alogorithms*. Moscow: Nauka. 304 pp. (In Russian)

Kolmogorov, A. N. 1992. *The Landholding in Novgorod in the 15th Century* (with comments by L. A. Bassalygo). Moscow: Nauka

Kolmogorov, A. N., Dmitriev, N. A. 1947. Branching stochastic processes. *Dokl. Akad. Nauk SSSR* 56: 7–10

Lebedev Physics Institute 1991. *Andrei Sakharov: Facets of a Life*. Collection of reminiscences prepared by P. N. Lebedev Physics Inst., USSR Acad. Sci. Gif-sur-Yvette: Editions Frontières. 730 pp.

Meneveau, C., Sreenivasan, K. R. 1991. The multifractal nature of turbulent energy dissipation. *J. Fluid Mech.* 224: 429–85

Millikan, C. B. 1939. A critical discussion on turbulent flows in channels and circular tubes. *Proc. 5th Intern. Congr. Appl. Mech.*, pp. 386–92. Cambridge, Mass.

Millionshchikov, M. D. 1939. Decay of homogeneous isotropic turbulence in a viscous incompressible fluids. *Dokl. Akad. Nauk SSSR* 22: 236–40

Millionshchikov, M. D. 1941. Theory of homogeneous isotropic turbulence. *Dokl. Akad. Nauk SSSR* 22: 241–42; *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.* 5: 433–46

Monin, A. S. 1950. Dynamic turbulence in the atmosphere. *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz.* 14: 232–54

Monin, A. S., Obukhov, A. M. 1954. Basic turbulent mixing laws in an atmospheric surface layer. *Tr. Geofiz. Inst. Akad. Nauk SSSR* 24 (151): 163–87

Monin, A. S., Yaglom, A. M. 1971. *Statistical Fluid Mechanics*, Vol. 1. Cambridge: MIT Press

Monin, A. S., Yaglom, A. M. 1975. *Statistical Fluid Mechanics*, Vol. 2. Cambridge: MIT Press

Novikov, E. A. 1971. Intermittency and scale similarity of the structure of turbulent flow. *Prikl. Mat. Mekh.* 35: 266–77

Obukhov, A. M. 1941a. Sound scattering in a turbulent flow. *Dokl. Akad. Nauk SSSR* 30: 611–14

Obukhov, A. M. 1941b. Spectral energy distribution in a turbulent flow. *Dokl. Akad.*

Nauk SSSR 32: 22–24; *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz.* 5: 453–66
Obukhov, A. M. 1942. On the theory of atmospheric turbulence. *Izv. Akad. Nauk SSSR Ser. Fiz.* 6: 59–63
Obukhov, A. M. 1949. Structure of the temperature field in a turbulent flow. *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz.* 13: 58–69
Obukhov, A. M. 1962a. Some specific features of atmospheric turbulence. *J. Geophys. Res.* 67: 311–14
Obukhov, A. M. 1962b. Some specific features of atmospheric turbulence. *J. Fluid Mech.* 13: 77–81
Prandtl, L. 1945. Über ein neues Formelsystem für die ausgebildete Turbulenz. *Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl.*, pp. 6–19
Richardson, L. F. 1926. Atmospheric diffusion shown on a distance-neighbour graph. *Proc. R. Soc.London Ser. A* 110: 709–37
Shiryayev, A. N. 1989. Kolmogorov: Life and creative activities. *Ann. Probab.* 17: 866–944
Shiryayev, A. N., ed. 1992. *Probability Theory and Mathematical Statistics.* Dordrecht: Kluwer. 598 pp.
Sossinsky, A., ed. 1992. *Information Theory and the Theory of Alogorithms.* Dordrecht: Kluwer
Spalding, D. B. 1991. Kolmogorov's two-equation model of turbulence. In *Turbulence and Stochastic Processes: Kolmogorov's Ideas 50 Years On*, pp. 211–14. London: Royal Society
Taylor, G. I. 1935. Statistical theory of turbulence, I-IV. *Proc. R. Soc. London Ser. A* 151: 421–78
Tikomirov, V. M., ed. 1991. *Mathematics and Mechanics.* Dordrecht: Kluwer. 552 pp.
Vishik, M. I., Fursikov, A. V. 1988. *Mathematical Problems of Statistical Hydrodynamics.* Dordrecht: Kluwer
von Kármán, T., Howarth L. 1938. On the statistical theory of isotropic turbulence. *Proc. R. Soc. London Ser. A* 164: 192–215
von Mises, R. 1941. Some remarks on the laws of turbulent motion in tubes. In *T. von Kármán Anniversary Volume*, pp. 317–27. Pasadena: Calif. Inst. Technol. Press
Yaglom, A. M. 1948. Homogeneous and isotropic turbulence in a viscous compressible fluid. *Izv. Akad. Nauk SSSR Ser. Geogr. Geofiz.* 12: 501–22
Yaglom, A. M. 1949. Local structure of temperature field in a turbulent flow. *Dokl. Akad. Nauk SSSR* 69: 743–46
Yaglom, A. M. 1981. Laws of small-scale turbulence in atmosphere and ocean. (In commemoration of the 40th anniversary of the theory of locally isotropic turbulence.) *Izv. Akad. Nauk SSSR Ser. Fiz. Atmos. Oceana* 17: 1235–57 (pp. 919–35 in English ed.)
Yaglom, A. M. 1990. Canonical correlation analysis and its meteorological applications. *Izv. Akad. Nauk SSSR Ser. Fiz. Atmos. Oceana* 26: 1248–66
Yanin, V. L. 1988. Kolmogorov as an historian. *Usp. Mat. Nauk* 43(6): 189–95
Zagustin, A. I. 1938. Equations for the turbulent motions of fluids. *Tr. Voronezh. Univ.* 10: 7–39