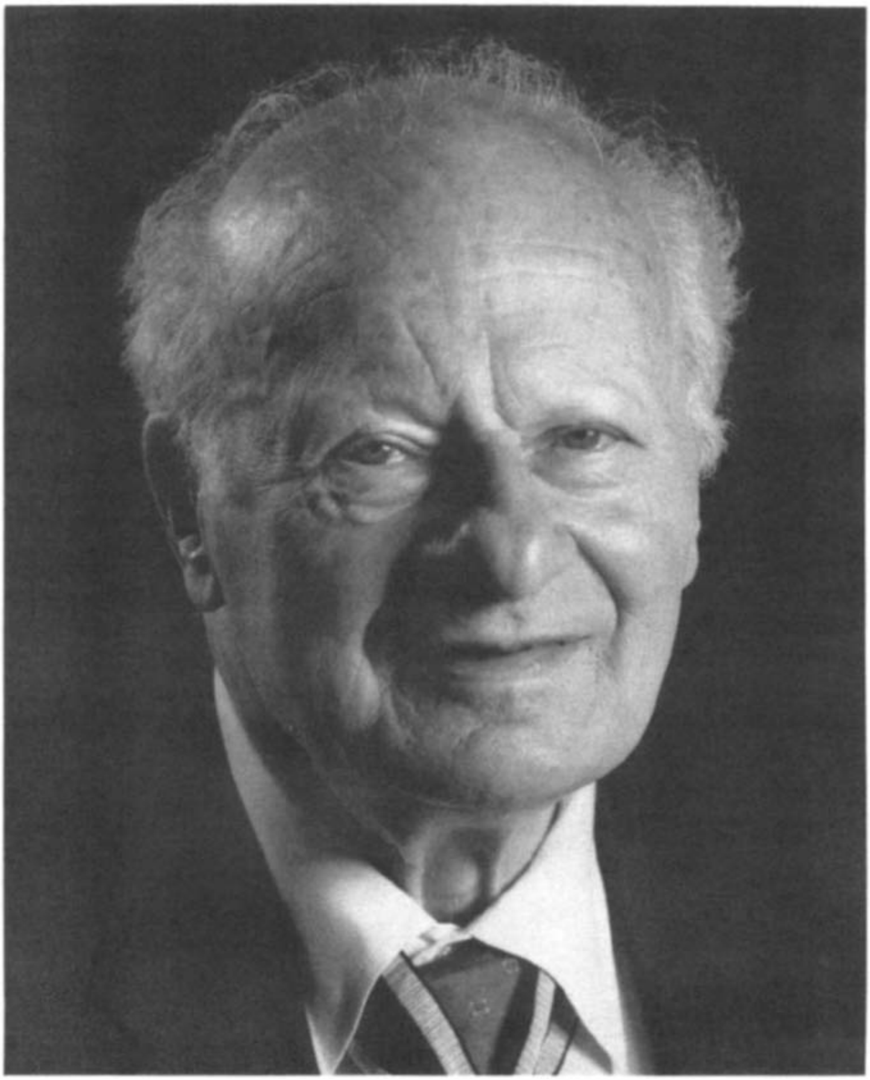

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NUCLEAR PHYSICS NEEDED FOR THE THEORY OF SUPERNOVAE

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CONTENTS

1. INTRODUCTION

Massive stars ($M > 8M_{\odot}$, where $M_{\odot}$ is the mass of the sun) develop into supernovae of Type II at the end of their lives. In the core of such massive stars, nuclear reactions occur involving increasingly heavy elements. The first reaction is of course the combination of protons into alpha particles; next, alpha particles combine into ^{12}C nuclei. As the star evolves, the core gets hotter and hotter, and reactions occur between two C nuclei, then Ne, then O + O, and finally between nuclei of Si. The result is matter consisting of nuclei of Fe and similar elements, which lie near the maximum of nuclear binding energy.

Once this stage is reached, no further energy can be released by nuclear reactions. There is still some pressure, generated by the degenerate electron gas in the core. The core then is similar to a white dwarf star. This stable

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configuration comes to an end, however, when the mass of the core reaches the Chandrasekhar limit. That limit is $1.44M_{\odot}$ if the nuclei contain equal numbers of neutrons and protons, and if the temperature is zero. If $Z < N$, the Chandrasekhar mass is smaller, but if the temperature is not zero, the critical mass is larger. In the core of massive stars, both of these deviations occur, and which of them predominates depends on the details of the stellar evolution preceding the supernova stage.

The mass of the core grows as the reaction Si $\rightarrow$ Fe proceeds; this usually takes one to a few days. Once the mass exceeds the Chandrasekhar limit, the core becomes unstable and collapses under gravity. The main part of this collapse takes about one second. The core, or at least its inner part, reaches very high densities, greater than the normal density of a nucleus, and thereby develops considerable nuclear pressure. This pressure then causes a rebound in which the parts of the star outside the core, i.e. the mantle and the envelope, are accelerated outward and ejected. This ejection is the main feature of the supernova phenomenon.

To describe the collapse of the core, one needs a good equation of state (EOS). The pressure and energy are dominated by those of the degenerate electron gas, but the nuclear fraction contributes an important part. Some of the nuclear material is in clusters resembling nuclei, but of very unusual ratio of neutrons to protons, and generally of very high atomic weight, perhaps between 100 and 1000. A moderate fraction of the nuclear material is present as a gas of neutrons and alpha particles, having much smaller density than the nuclei floating in it. The first task therefore is to establish an EOS for these conditions.

Already in the presupernova evolution, the nuclei tend to capture electrons and thereby grow richer in neutrons. The fraction of protons in the nucleus is generally denoted by x, so $x < 1/2$. In this presupernova stage, x is equal to Y_e, the number of electrons per nucleon.

During the collapse phase, further electron capture takes place, so Y_e decreases further. Electron capture may take place on free protons or on complex nuclei. It is therefore necessary to know the fraction X_H of free protons among all the nucleons; this depends chiefly on the temperature. For the capture by complex nuclei, we need to know the mass differences between isobars; this depends mainly on the nuclear symmetry energy. The nuclei involved are generally of atomic weight $A = 70$ to 100. We also need to know the rate of electron capture.

When the central core is compressed to densities greater than normal nuclear matter density, $\rho_s = 0.16\ \mathrm{fm}^{-3}$ (nuclear density of $1\ \mathrm{fm}^{-3} = 1.67 \times 10^{15}\ \mathrm{g\ cm}^{-3}$), it is necessary to know the EOS at these supernuclear densities. This part of the problem is probably the most difficult as we have no experimental data to guide us. At the same time, it is the most important because on this EOS depends the question of whether or not the shock resulting from the compression of the central core will actually expel mantle and envelope of the star. This point is still controversial, but we at least know that supernovae do occur.

In the shock propagating outward from the center, the EOS is much simpler. The entropy and temperature are high in this case, and the density is relatively low. Therefore we have free nucleons, interacting weakly.

Entropy plays a dominant role in the supernova theory. During the collapse, the entropy of each element of matter stays nearly constant. Entropy is most conveniently given as the entropy per nucleon, in units of k_B, the Boltzmann constant. The entropy resulting from presupernova calculations is generally of the order of one in these units. By contrast, the entropy behind the shock wave is usually of the order of 10 units.

The presupernova evolution, as already mentioned, makes nuclei neutron rich; $x = 0.42$ is a typical value. As more electrons are captured during the collapse, x decreases. However, this decrease comes to an end when the density of matter reaches a value of about 10^{12} g cm^{-3}. The reason is that neutrinos are then trapped as a result of scattering by the nuclei, owing to the neutral weak interaction. Once this trapping occurs, the lepton fraction,

$$Y_L = Y_e + Y_\nu, \quad 1.$$

remains constant.

Figure 1 shows the density distribution in the star before collapse. Densities up to about 10^9 occur. The places are marked at which various nuclear reactions occur and the relevant temperature given, in units of 10^9 K. After collapse, higher densities and temperatures are reached.

Of course, knowledge of the EOS is only the first step. Knowing the EOS, one must then compute the hydrodynamics of the star, both in the collapse phase and in the subsequent reexpansion.

2. BASIC EQUATION OF STATE

The basic equation of state (EOS) has been derived by Lamb et al (1–4). Matter consists of a nuclear and an electron component. Neutrinos have negligible interaction and can be disregarded as long as they escape from the star, i.e. for densities $\rho < 5 \times 10^{11}$ g cm^{-3}. The state of matter is described by the density ρ, the temperature T, and Y_e, the number of electrons per nucleon. For the temperture, we generally give kT in MeV; 1 MeV $= 1.16 \times 10^{10}$ K.

The forces are nuclear and electromagnetic; statistical mechanics is used to derive the EOS. Because of the attractive nuclear forces, the nuclear

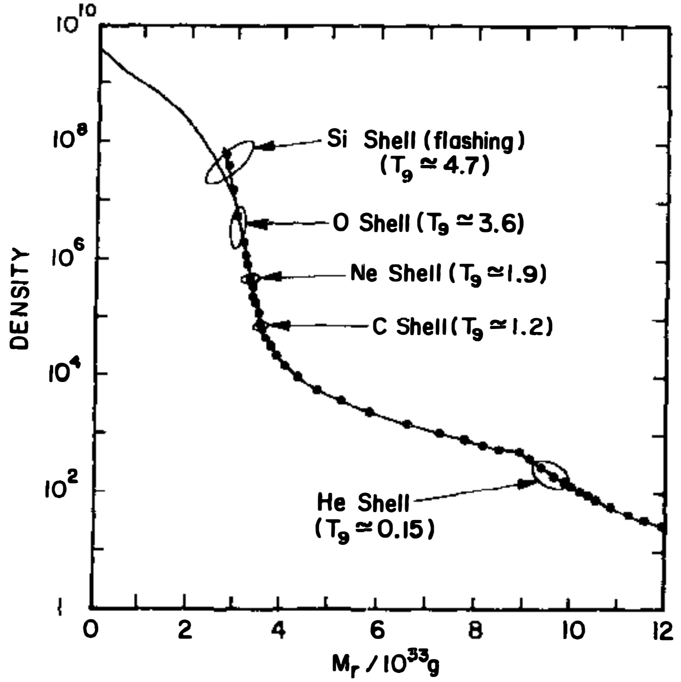

Figure 1 Density distribution before supernova collapse. The abscissa represents the enclosed mass. The temperatures are noted at which various nuclear reactions take place, in units of 10^9 K. According to Arnett, W. D., *Astrophys. J.* 218: 815 (1977).

component condenses into nuclei. Outside of these, we have a lower density "gas" of nucleons and alpha particles. Thus we have a two-phase system; the chemical potential μ must be continuous at the boundary between the two phases, both for neutrons and protons.

The electrons are essentially uniformly distributed in space, both inside and outside the nuclei. In equilibrium, we must have

$$\mu_e = \mu_n - \mu_p = \hat{\mu}. \tag{2}$$

Especially at the higher temperature, positrons are present; their chemical potential is $\mu_+ = -\mu_- (= -\mu_e)$.

The size of the nuclei is determined by the combined action of surface and Coulomb energy. The surface energy was derived by Ravenhall, Bennett & Pethick (5) and can be written as

$$W_{\text{surf}} = A^{2/3} 290 x^2 (1-x)^2, \tag{3}$$

where x is the fraction of protons inside the nucleus. In most cases when there are not many nucleons outside nuclei, $x = Y_e$. All nuclei are assumed to have the same A and Z; assuming a statistical distribution does not make much difference.

At high temperature, the nuclei evaporate into a gas of nucleons, as described in detail by Lamb et al (1–4). There is a critical point at $T \approx 15$ MeV.

As the density increases to about one half of the saturation nuclear density, we no longer have nuclei, but instead all of space is filled by nuclear matter of uniform density, with empty bubbles distributed in it. This phase is commonly called "Swiss cheese." Ultimately, these empty bubbles disappear and we get nuclear matter filling space uniformly. This is discussed in more detail in Section 4. The saturation density of symmetric nuclear matter we take to be

$$\rho_s = 0.16 \text{ fm}^{-3}. \tag{4}$$

Because of their Coulomb repulsion, the nuclei form a lattice, and we assume the same for the bubbles. It is often convenient to use a Wigner-Seitz cell, which contains one nucleus. For calculations, that cell is usually assumed spherical.

For definiteness, Lamb et al (1–4) use a specific nuclear force, namely the Skyrme force I. This is somewhat too stiff, having a compression modulus $K = 370$ instead of 220 MeV, which is at present the best value. However, a definite nuclear force is useful to calculate unique values of several physical quantities: the entropy S, measured in units of k_B per nucleon; the chemical potentials μ_n and μ_p of neutrons and protons; the fraction X_H of heavy nucleons that are in nuclei; the pressure P in MeV fm^{-3} (one of these units is 1.6×10^{33} dyne cm^{-2}); the energy per nucleon in MeV; and the free energy $F = E - TS$.

Using such calculations, Figure 2 shows the adiabats $S = 1$ to 5 for $Y_e = 0.35$, which is close to the Y_e relevant in calculations of the supernova collapse. Also shown are the curves for $X_H = 0.5$ and 0.1; X_H depends chiefly on μ_n and ρ. From calculations of the evolution of stars before supernova collapse, one finds that S is about 1. Figure 2 shows then that generally X_H is near 1, i.e. only a small fraction of the nuclear material is in the "gas."

The pressure from the nuclear phase is calculated. It is often negative because the Coulomb energy per nucleon is smaller than it is for free nuclei. This in turn is due to the fact that some of the electrons are inside nuclei. The total pressure is therefore mostly due to electrons. When the material has overall density $\rho_s = 0.16$ fm^{-3} and $Y_e = 0.3$, the electron contribution is

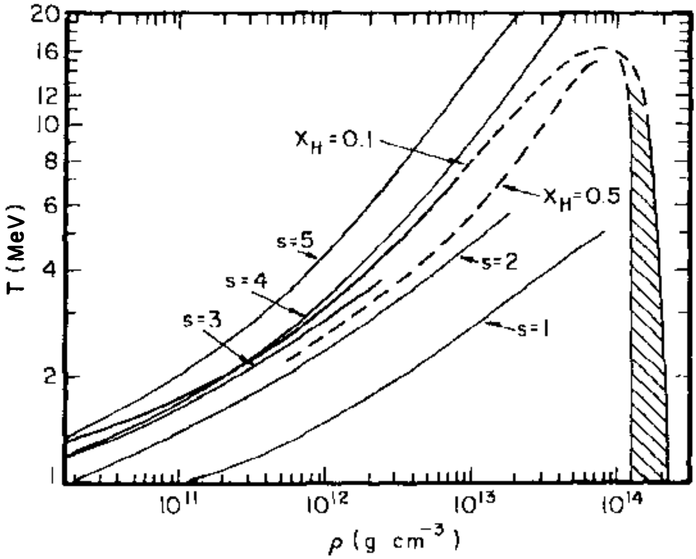

Figure 2 Adiabats of nuclei plotted against temperature and density. The number of electrons per nucleon is $Y_e = 0.35$. The dashed curve is $X_H = 0.5$, where half the nucleons are in nuclei and the other half are free. In the shaded region, we have uniform nuclear matter with empty bubbles in it. From Lamb et al (1).

$$P_e/\rho = 20 \text{ MeV}. \tag{5}$$

By comparison, the nuclear P/ρ is the order 1 MeV.

For many problems of the dynamics of the supernova, the adiabatic index plays an important role,

$$\Gamma = (\partial \log P/\partial \log \rho)_S. \tag{6}$$

Figure 3 shows Γ as a function of ρ. There is no resistance to the gravitational collapse of the star as long as $\Gamma < 4/3$. It is clear from the figure that resistance will not occur until $\rho \gtrsim \rho_s$. Once this density is reached, Γ becomes large (see Section 6).

The fact that $\Gamma < 4/3$ for the material with $S \approx 1$ shows that the collapse of supernovae cannot be stopped until the material reaches nuclear density. This was recognized in (1–4) and also elaborated by Bethe et al (6). It was contrary to the belief held by astrophysicists before 1978.

The energy in nuclei increases with T. This is due to excitation of the higher energy levels of the nucleus. The density of energy levels increases exponentially with the entropy S in nuclei. Lamb et al (1–4) described this effect in terms of a fermi gas of nucleons inside the nucleus. Experimentally,

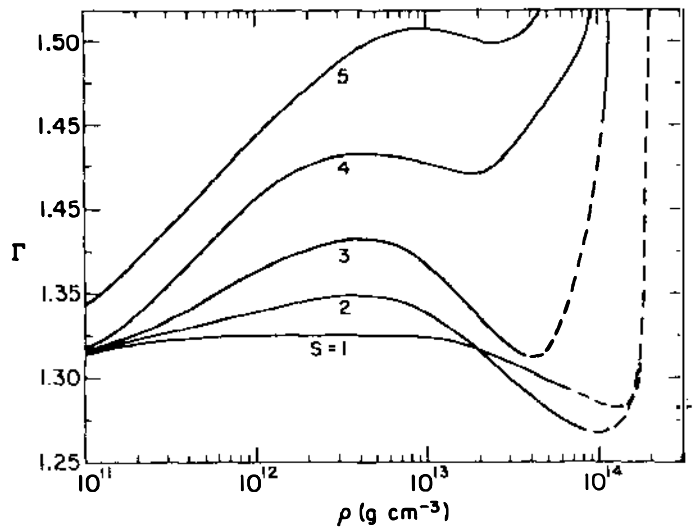

Figure 3 Adiabatic index, $\Gamma = (\partial \log P/\partial \log \rho)_S$, vs density and temperature for various entropies S. From Lamb et al (1).

it is known (7) that the level density of excited levels of nuclei is greater than the fermi gas approximation. It is as if the entropy inside nuclei were multiplied by a factor

$$m^*/m > 1. \tag{7}$$

Some part of this increased level density is a surface effect. The energy in excitation of nuclei does not contribute to the pressure.

The difference $\hat{\mu} = \mu_n - \mu_p$ is positive because all the nuclei concerned have an excess of neutrons. The value of $\hat{\mu}$ depends on the nucleon symmetry energy, cf Sections 5 and 7. The quantity $\hat{\mu}$ is important for the fraction of protons, which is

$$Y_p = Y_n e^{-\hat{\mu}/T}. \tag{8}$$

Y_p gives the number of protons in the gas phase as a fraction of the total number of nucleons in gas plus nuclei. Because of the exponent, Y_p is generally very small, about $1\text{–}5 \times 10^{-4}$, see Table 4.

Lamb et al (3) discuss the phase transitions, from nuclei to bubbles, and then from bubbles to uniform nuclear matter. The change of properties at the phase transitions is generally small. The transition from bubbles to nuclear matter would in principle be first order but is in practice second order; the latent heat is only the lattice Coulomb energy, which is very small, especially because the empty bubbles are quite small before they disappear (see Section 4).

Reference (4) contains a large number of useful tables and figures. It also discusses the case of high densities, $\rho > 10^{12}$ g cm^{-3}, when neutrinos are trapped in the stellar material because of the weak neutral interaction. In this case, the total number of leptons per nucleon (Equation 1), not Y_e, is given a priori. The condition of chemical equilibrium then determines Y_e. Curves are given in (4) for given values of Y_L.

3. MORE ELABORATE EQUATIONS OF STATE

Ravenhall, Pethick & Wilson (7) discussed the shape of the nuclei and bubbles. At low overall density, the nuclei are spherical. At higher density, however, energy is minimized by choosing different shapes. At first, the spheres deform into prolate ellipsoids, arranged parallel to each other. Next, these merge into long cylinders. After some intermediate steps, the most favorable configuration is flat plates. At this stage, there is no difference between plates of nuclear matter in a dilute gas of nucleons, or plates of gas between uniform nuclear matter. With further increasing density, they become spaghetti-like gas spaces embedded in uniform nuclear matter, and finally spherical bubbles of gas in nuclear matter.

With these assumptions, Ravenhall et al got a rather smooth EOS, in particular the total nuclear energy per unit volume becomes nearly smooth as a function of density. In reality, they argue, the spheres should be deformed, and likewise the cylinders and plates should have bulges. This would make the EOS even smoother.

The maximum nuclear energy per unit volume turns out to be

$$0.015 \text{ MeV fm}^{-3}. \tag{9}$$

For comparison, the nuclear energy of nuclear matter at normal density is 2.5 MeV fm^{-3}. So the entire energy in the subnuclear density regime is quite small, about 0.6%. Ravenhall et al used the same Skyrme interaction as Lamb et al (1–4).

An alternative Skyrme interaction was used by Bonche & Vautherin (8). Their interaction gives the correct equilibrium density and the observed compression modulus, and also other quantities in agreement with observation. The interaction is taken from Bohigas et al (9). The characteristic quantities at nuclear density are as follows:

$$k_F = 1.33,\ K = 213,\ m^*/m = 0.79,\ \text{symmetry energy} = 30.6 \text{ MeV}. \tag{10}$$

Bonche & Vautherin solved the Hartree-Fock equation in a Wigner-Seitz cell. Just as in (1–4), they assumed spherical nuclei plus a low density nucleon gas at low overall density, and empty bubbles in nuclear matter for high density ρ. The calculations were done at various temperatures. Curves were given for P/ρ, the energy per particle, and the equilibrium atomic weight A as functions of the density for $T = 1$ and 4 MeV. In a table, they give T, P, and A along the adiabat $S = 1$ for $\rho = 0.02$ to $0.07\ \mathrm{fm}^{-3}$; A goes from 300 to 1000.

4. A SIMPLIFIED, ANALYTICAL EQUATION OF STATE

Cooperstein (10) constructed a simple EOS whose components can be understood physically. He had three reasons for constructing this EOS, namely (*a*) to get analytical formulae, (*b*) to get a smooth EOS that goes from low density to normal nuclear saturation density ρ_s without discontinuities, and (*c*) to use observed nuclear parameters, rather than some nucleon interaction, such as Skyrme, that may not give good agreement with experimental data.

Requirement (*b*) is essential to obtain stability in hydrodynamic calculations of the implosion of the supernova. Ravenhall et al (7) showed that a smooth EOS is actually obtained if the change of shape of the nucleus with increasing density is properly considered. Cooperstein obtains similar smoothness by finding a correct formula for spherical nuclei at low density and for spherical bubbles at high density, then interpolating sensibly between them. The result is Equation 18; it agrees almost precisely with (7).

We define the fraction of space occupied by nuclei,

$$u = \rho/\rho_0, \qquad 11.$$

where ρ is the overall average density and ρ_0 the density of the nuclei. We assume that the nuclei are compressible, as in (3), and write

$$\rho_0 = \rho_s\theta, \qquad 12.$$

where ρ_s is the saturation density. The latter is taken to depend on $x = Z/A$, the fraction of protons inside the nuclei, and is assumed to be

$$\rho_s(x) = 0.16\ \mathrm{fm}^{-3}\ \phi(x)$$
$$\phi(x) = [1-\tfrac{3}{4}(1-2x)^2]. \qquad 13.$$

This equation says that the saturation density becomes smaller for unsymmetric matter. Probably this effect is underestimated because Equation 13 gives $\phi(x) = 1/4$ for pure neutron matter while in reality such matter is not bound at all; Equation 13 is appropriate only for x near 1/2. At the boundary between nuclei and gas, the pressure is continuous, and so are the chemical potentials of neutrons and protons.

The energy per nucleon of bulk nuclear matter (in MeV) is given by

$$W_{\text{bulk}} = -16+29.3(1-2x)^2+\frac{1}{18}K(1-\theta)^2, \tag{14}$$

where K is the compression modulus and is assumed to be independent of x. The coefficients are chosen to give the best possible agreement with Bonche & Vautherin (8); $K = 220$ MeV. This value was obtained experimentally by Blaizot et al (11). The volume and bulk symmetry coefficients of -16 and 29.3 MeV are not important for the ensuing analysis (but the symmetry energy is important in Section 5 of this article). For finite nuclei, one has to add to W_{bulk} another term, W_{size} (12), which includes all surface and Coulomb effects; it is

$$W_{\text{size}} = \beta(x)\theta^{-1/3}G(u), \tag{15}$$

$$\beta(x) = 75x^2(1-x)^{4/3}\phi^{-1/3}. \tag{16}$$

The factor $\beta(x)$ is the characteristic surface and Coulomb energy of an isolated nucleus. W_{size} depends on the density of the nucleus as $\rho_0^{-1/3}$, hence $(\theta\phi)^{-1/3}$.

When the nucleus is at the center of a Wigner-Seitz cell, the Coulomb energy is diminished, as compared with a free nucleus, by a factor

$$g(u) = 1-\tfrac{3}{2}u^{1/3}+\tfrac{1}{2}u. \tag{17}$$

When the sum of surface and Coulomb energy is minimized, it becomes proportional to $g^{1/3}$. Numerical values are shown in Table 1. For u near 1, nuclei are replaced by empty bubbles in nuclear matter, and a somewhat different formula is obtained. Cooperstein now interpolates between these two cases with the formula

$$G(u) = (1-u)[g^{1/3}(u)+g^{1/3}(1-u)]. \tag{18}$$

This formula is correct for u near 0 and near 1, and reproduces very well the more elaborate calculations in (7).

Table 1 $G = g^{1/3}(u)$

u	G
1/8	0.721
1/4	0.565
1/2	0.406

The same calculation gives for the mass of the average nucleus

$$A = 192(1-x)^2(\theta\phi)^{-1}g^{-1}(u). \qquad 19.$$

At $x = 1/3$, which is standard for nuclei in a collapsing supernova, $A = 85$ for $u = 0$; it increases to 225 for $u = 1/3$ and to 1100 for $u = 1/2$.

The energy

$$W = W_{\text{bulk}} + W_{\text{size}} \qquad 20.$$

depends on the density both through u and θ. The pressure, by general thermodynamics, is given by

$$\frac{P}{\rho} = \frac{\mathrm{d}W}{\mathrm{d}\ln\rho} = \frac{\partial W}{(\partial\ln\theta)_u} = \frac{\partial W}{\partial\ln u_\theta}. \qquad 21.$$

This equation gives the pressure, and in addition gives a relation between θ and u, namely

$$\theta^{4/3}(1-\theta) + \varepsilon_{\text{size}}\tfrac{1}{3}(G+G') = 0, \qquad 22.$$

where the prime denotes d/d ln u. This equation insures pressure balance at the surface, and

$$\varepsilon_{\text{size}} = 9\beta/K \qquad 23.$$

can be calculated as a function of x.

This procedure gives perfectly reasonable values for θ as long as $x < 1/3$. But for nearly symmetric nuclear matter and u near 1, there is no solution: in this case the bubble in nuclear matter would collapse.

This trouble is resolved if we consider matter at nonzero temperature, the only case of physical interest. For finite temperature we write the free energy

$$F = E - TS = W_{\text{bulk}} + W_{\text{size}} - \frac{a}{A}\frac{m^*}{m}T^2. \qquad 24.$$

In the thermal term, Cooperstein separates the dependence on the simple fermi gas level density parameter

$$\frac{a}{A} = \frac{\pi^2}{4\varepsilon_F} = [(\theta\phi)^{2/3} \times 14.9 \text{ MeV}]^{-1}, \qquad 25.$$

from an effective mass m^*/m. The entropy is given by

$$S = -\frac{\partial F}{\partial T} = 2\frac{a}{A}\frac{m^*}{m}T. \qquad 26.$$

This entropy is closely related to the level density of excited states of the nucleus, which is e^S. With $m^*/m = 1$, we recover the level density for the simple fermi gas.

It is well known from measurements (13) that the level density in actual nuclei is greater than that given by the fermi gas. In fact, for nuclei in the Fe region, $m^*/m = m_0 \approx 2$. This large number, and hence large level density, is at least partly due to the surface of the nucleus. On the other hand, in a saturated uniform medium, the surface effects disappear and[1]

$$m^*/m = m_s \approx 0.7, \tag{27.}$$

where the last number is derived from Brueckner theory. Cooperstein now adopts an interpolation

$$\frac{m^*}{m} = m_s + (m_0 - m_s)\frac{W_{size}}{W_{size}(^{56}\mathrm{Fe})}. \tag{28.}$$

This is reasonable because m^*/m is an effect of the surface of the nucleus. However, the actual calculation of m^*/m is a rather difficult task. Bonche et al (14) have considered this problem in a finite-temperature Hartree-Fock theory for ^{208}Pb and ^{56}Fe.

The entropy (Equation 26), as mentioned, is related to the level density and thus to nuclear excitation. Similarly, the energy and free energy here considered are those in nuclear excitation.

After some algebra, Cooperstein finds that θ is made smoother by the inclusion of finite temperature. At $u = 0$ and $x = 0.4$, $\theta \approx 1.05$ for any value of the entropy. Near $u = 1$, at $S = 0$, θ becomes very low and the bubble (recall that for $u > 1/2$ we have bubbles in nuclear matter) would collapse. However, for $S \geq 1$, this is not true; the bubbles are saved by finite temperature. If the entropy in nuclear matter is $S = 1$, the expression for θ is very smooth and is approximately

$$\theta = 0.7 + 0.02T. \tag{29.}$$

The EOS of the gas in the low density phase is relatively simple. The pressure of the gas further compresses the nuclei, i.e. increases θ somewhat. More important, since the low density phase consists mostly of neutrons, its existence makes the high density phase less neutron rich. Both effects were taken into account in the actual calculations.

The transition from nuclei to "Swiss cheese" is completely smooth, by the definitions of the theory. The transition from "Swiss cheese" to uniform nuclear matter is also very smooth, as Ravenhall et al (7) pointed out. If the Coulomb interaction between bubbles is neglected, this transition is

[1] Modern calculations give $m^*/m = 0.85$, cf Section 6.

second order. In fact, the Coulomb interaction is very small because near the transition the bubbles are small and fairly widely separated.

To make this transition smooth, Cooperstein uses a shortcut: he continues the bubble solution up to $u = 1$, and then uses uniform matter at higher ρ. This makes the transition exactly second order. At $u = 1$ and $Y_e = 0.3$,

$$\frac{P - P_{\text{lep}}}{\rho} \approx -0.7 \text{ MeV}. \tag{30.}$$

For $S = 1$, the transition to uniform nuclear matter is about at $T = 5$ MeV, hence from Equation 29 we find $\theta = 0.8$.

This last result indicates that, after the elimination of bubbles, the uniform nuclear matter does not have saturation density but is actually stretched to 80% of that density. This means that, at this point, the nuclear pressure is negative. According to Equation 30, at the transition point the nuclear P/ρ is -0.7 MeV, which compares with a pressure in the degenerate electron gas of

$$P_{\text{lep}}/\rho \approx 20 \text{ MeV}. \tag{31.}$$

If the entropy is higher than 1, the nuclear pressure remains positive all the time. Collapsing supernovae have nearly $S = 1$.

Together with the nuclear pressure, shown in Figure 4 as a function of density for different entropies, the adiabatic index behaves very smoothly. While it goes down to about 1.27 in the work of Lamb et al (4), its minimum in Cooperstein's theory is 1.29 for $S = 1$ or 2; for $S = 1.5$, it is about 1.30. Table 2 gives the variation of various important quantities for $\rho = 0.001$ to 0.1 nucleon per fm^3 corresponding to 1.7×10^{12} to 1.7×10^{14} g cm^{-3}.

The most important quantity for supernova collapse is P/ρ. It is remarkable that, after all the effort to construct a nuclear equation of state, the nuclear contribution to P/ρ is very small compared to that of the degenerate electrons; this ensures the smoothness of P vs ρ. The fraction of nuclear material in the gas phase, $1 - X_H$, is small, and actually decreases with increasing density, from 15 to 2%—somewhat against intuition. The density of the nuclei, ρ_0, varies appreciably, from 0.157 to 0.127 fm^{-3}, which shows that one could not obtain a valid EOS by assuming the nuclei incompressible. The temperature, at constant $S = 1$, rises substantially with density, from $T = 1$ to 6 MeV; T is important for electron capture (see Section 5).

5. CAPTURE OF ELECTRONS

During the collapse of the supernova, the density of each material element increases. When the density exceeds about 10^{10} g cm^{-3}, electrons will be

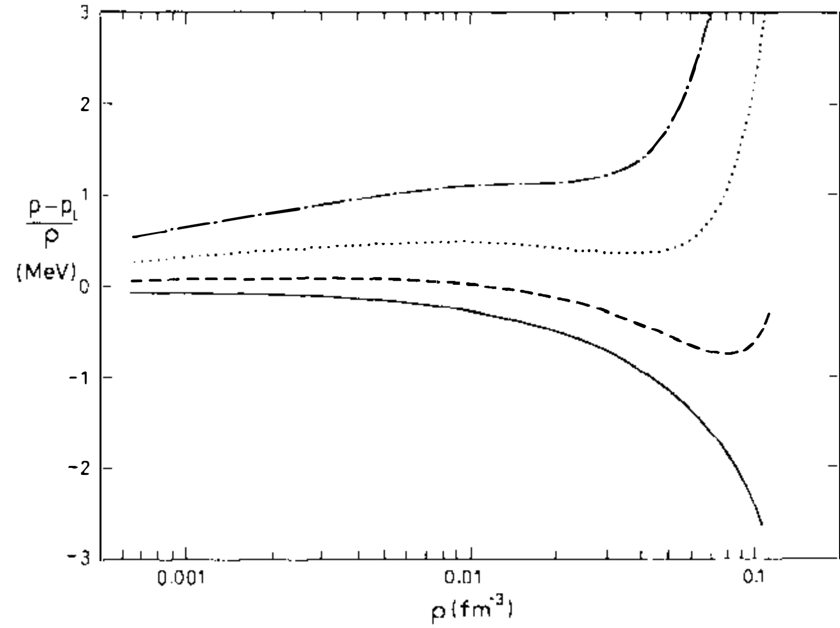

Figure 4 Pressure due to nuclear component vs density. The density is given in nucleons per fm^3. Solid line $S = 0.5$, dashed $S = 1$, dotted $S = 1.5$, dash-dot $S = 2$. From Cooperstein (10).

captured, converting protons into neutrons. When the density becomes greater than about 10^{12} g cm^{-3}, neutrinos are trapped in the material of the star, and this puts an end to electron capture.

The cross section for the fundamental process, capture of electrons by protons, is given by the universal Fermi theory,

Table 2 Physical quantities as functions of density, for entropy $S = 1$ and $Y_e = 0.3$ [from Cooperstein (10)]

ρ (fm^{-3})	Γ	$\frac{P-P_e}{\rho}$ (MeV)	ρ_0 (fm^{-3})	T (MeV)	$1-X_H$
0.001	1.332	0.1	0.157	1.3	0.13
0.003	1.327	0.1	0.154	1.7	0.13
0.01	1.311	0.0	0.149	2.6	0.12
0.03	1.293	−0.3	0.140[a]	3.7	0.07
0.1	1.46	−0.6[a]	0.141	6.2	0.01

[a] Minimum values.

$$\sigma(e \to \nu) = 4.5 \times 10^{-44}\,\varepsilon^2\ \mathrm{cm}^2, \tag{32.}$$

where ε is the neutrino energy in MeV. In the early literature, it was assumed that electrons are captured only by free protons. The number of free protons, however, is very small; it is related to the number of free neutrons by

$$X_p = X_n e^{-\hat{\mu}/T}, \tag{33.}$$

where X_n is the fraction of free neutrons, which is nearly equal to $1 - X_H$ (Table 2) and hence itself is fairly small. The temperature T is typically 1 MeV while the chemical potential $\hat{\mu}$ is 5–10 MeV. Numerical values of the fraction of free protons are given in Table 4; the fraction is of order 10^{-4} to 10^{-3}.

Because of the small concentration of free protons, Bethe et al (6) suggested that the capture takes place on complex nuclei. Using the shell model, they assumed that the protons are generally in the $f_{7/2}$ shell, and that they can have an allowed transition of the Gamow-Teller type to the empty $f_{5/2}$ neutron shell. In this manner, they obtained strong electron capture and found that after collapse the electron fraction Y_e will be quite small.

Fuller (15) pointed out that the $f_{5/2}$ shell of neutrons is filled at $N = 38$. The initial Y_e is about 0.44 (see Table 3), and at a density $\rho = 10^{10}\ \mathrm{g\ cm}^{-3}$, the expected mass number is already $A = 68$; so the neutron number is just about 38. From this point on, the $f_{5/2}$ shell of neutrons is no longer empty, and therefore the Gamow-Teller transition from the $f_{7/2}$ shell of the

Table 3 Electron fraction and entropy before collapse, as a function of enclosed mass of the star, M_r[a]

M_r	Y_e	S
0	0.422	0.63
0.25	0.430	0.70
0.52	0.441	0.79
0.77	0.447	1.12
1.02	0.461	1.42
1.28	0.468	1.68

[a] M_r is given in units of the mass of the Sun, $M_\odot$. The mass of the Fe core is 1.28; outside of the core are elements such as Si, S, and O, for which $Y_e = 0.500$. From Woosley, S. E., Calculation of stellar evolution of a star consisting only of He and heavier elements, of mass $4M_\odot$, without semiconvection and convective overshoot (private communication, 1987).

protons is no longer allowed. Consequently there will be much less electron capture than assumed in (6).

Epstein & Pethick (17) pointed out that the final Y_e and S are very sensitive to various parameters. In turn, the subsequent development of the supernova is sensitive to Y_e and S.

The most complete treatment of electron capture to date has been given by Cooperstein & Wambach (18). They find that it is very important that the temperature T is not zero but of the order 1 MeV. This means that protons can be excited to higher shells, and that there can be vacancies in certain neutron shells. They chose as a typical nucleus ^{82}Ge, having 32 protons and 50 neutrons. The protons occupy the shells $1f_{7/2}$ and $2p_{3/2}$; very close above the fermi energy is $1f_{5/2}$. The neutrons occupy all shells through $1g_{9/2}$, above this is a large gap to $2d_{5/2}$. They calculate

$$\hat{\mu} = \mu_n - \mu_p \approx 8 \text{ MeV}. \tag{34}$$

They consider both allowed and first-forbidden transitions. Among the latter are

$$\begin{aligned}&(p)2p_{3/2} \rightarrow (n)2d_{5/2}, 3s_{1/2}, 2d_{3/2}\\ &(p)1f_{7/2} \rightarrow (n)2d_{5/2}, 1g_{7/2}, 2d_{3/2}.\end{aligned} \tag{35}$$

With $T = 1$ or 1.5 MeV, allowed transitions become possible again. The most important of these turns out to be

$$(1g_{9/2})_p \rightarrow (1g_{7/2})_n, \tag{36}$$

but others also contribute substantially. This is again a Gamow-Teller transition. Here one must take into account that such transitions are strongly suppressed, as was shown theoretically by Ejiri (19). His prediction is well confirmed by p-n reactions carried out with high energy protons; these experiments indicate that the matrix element is suppressed by a factor $\gamma \approx 1/3$.

Using these arguments, Cooperstein & Wambach (18) calculate the various allowed and forbidden transitions. For definiteness, they take an average between ^{82}Ge and ^{90}Zr. They then calculate the manner in which Y_e decreases as a function of increasing density, using standard dynamics of supernova collapse. They also calculate other relevant dynamic quantities; these are given in Table. 4.

The table shows that the temperature increases only very slowly, from 0.9 to 1.5 MeV. For Y_e they assume an initial value of 0.42, which is typical for the center of a star at the end of its evolution. Y_e decreases rapidly after $\rho > 10^{11}$ g cm^{-3}. (It has been assumed that the neutrino levels are all empty, i.e. that there is no trapping.) Accordingly, $\hat{\mu}$ increases rapidly, and

Table 4 Properties of matter during collapse, as a function of density [from Cooperstein & Wambach (18)][a]

$\log \rho$	Y_e	T	$\hat{\mu}$	μ_e	S	$10^4 X_p$	$1-X_H$ (%)	Z	A
10	0.420	0.885	3.1	8.1	1.00	2.7	3.6	28	68
10.5	0.418	1.04	3.3	11.9	0.99	5.4	4.8		
11	0.410	1.21	4.5	17.5	0.97	5.3	4.7		
11.5	0.389	1.37	7.4	25	0.93	1.6	4.2		
12	0.358	1.49	12.1	36	0.91	0.16	5.9	37	97

[a] T, $\hat{\mu}$, and μ_e are in MeV; ρ is in g cm^{-3}; S is in units of k_B. Z and A are average values. The Ejiri factor on the Gamow-Teller matrix element is assumed to be 1/3.

the fraction of free protons in the material decreases for $\rho > 10^{11}$ g cm^{-3}. The total fraction of nucleons that is not in heavy nuclei, $1-X_H$, stays approximately constant. The average nuclear charge and mass number increase with density (see Equation 19). The Y_e at $\rho = 10^{12}$ g cm^{-3} should not be taken seriously because before this density is reached neutrinos begin to get trapped, keeping at least the total lepton fraction Y_L, Equation 1, higher.

The entropy S, assumed to be 1.0 to start with, decreases slightly. Bethe et al (6) had already shown that such a decrease will occur if the electrons are captured by free protons. However, it came as a surprise that S also decreases in capture by complex nuclei. The reason is that, because of the elevated temperature, most transitions start from a nucleus with either a proton in an excited state or a hole in a neutron shell that would be fully occupied at $T = 0$. Table 5 gives the energy that is transferred to the nucleus by electron capture,

$$\varepsilon^* = (\varepsilon_n - \mu_n) - (\varepsilon_p - \mu_p) - M_{pn}, \tag{37.}$$

Table 5 Average energies of outgoing neutrinos E_ν and of nuclear excitation ε^* (in MeV) and final electron fraction Y_{ef} from electron capture process, for two densities [from Cooperstein & Wambach (18)]

Quantity	$\log \rho$	Electron capture process			
		free protons	allowed	forbidden	average
E_ν	10	7	6	5	6
E_ν	12	30	24	19	22
ε^*	10		−1.5	0	
ε^*	12		−5	+1	
Y_{ef}		0.38			0.36

where M_{pn} is the mass difference between neutron and proton, 1.29 MeV. Table 5 shows that this ε^* is negative on average for all allowed transitions.

Table 4 shows that the chemical potential of the electrons, μ_e, is substantially greater than $\hat{\mu}$. The difference is mostly given as kinetic energy to the outgoing neutrino. Table 5 gives the average energy of that neutrino for the initial and the final density of matter. It gives the E_ν for capture on free protons, for allowed capture on complex nuclei, and for forbidden transitions. It also gives the average of these capture processes. The table shows that E_ν increases with ρ, as would be expected, and that it is largest for capture on free protons.

For $\rho < 10^{11}$ g cm^{-3}, the capture on free protons dominates. Around 10^{11}, first-forbidden transitions become important and later allowed transitions. By $\rho = 10^{12}$ g cm^{-3} these two types of transitions are equal and about three times the capture on free protons.

Table 5 also gives the final electron fraction, Y_{ef}. Surprisingly, the quenching of the Gamow-Teller transitions according to Ejiri (19) does not make a great deal of difference in Y_{ef}.

The relatively high energy of the emitted neutrinos means that these are easily trapped. But even after neutrino trapping, electron capture continues until equilibrium is established by electrons and neutrinos, i.e.

$$\mu_e - \mu_\nu = \hat{\mu} - M_{pn}. \qquad 38.$$

In this capture process, only electron neutrinos are produced. Later on in the evolution of the supernova, other flavors of neutrinos are also emitted. But this occurs by the plasma process,

$$e^+ + e^- \rightarrow \nu + \bar{\nu}, \qquad 39.$$

a process not reviewed here.

6. HIGH DENSITY EQUATION OF STATE

The standard equation of state for densities greater than ρ_{nm} (normal nuclear matter density, called ρ_s in previous sections) is discussed by Friedman & Pandharipande (20). They use a rather accurate potential between two nucleons, and calculate the energy per particle for densities up to $9\rho_{nm}$. Their EOS is rather stiff, i.e. the energy and hence the pressure increase rapidly with density.

At these high densities, the nucleons become to some extent relativistic. Therefore it is believed that Dirac theory should be used to treat the nuclear matter problem. This is also indicated by experiments on the scattering of nucleons by nuclei in the laboratory. It has been found that especially the polarization and the spin rotation parameter are much better described by a Dirac theory than by Schrödinger wave functions (20a).

Once relativistic theory is used, it is necessary to distinguish between the ordinary density

$$\rho = \bar{\psi}\gamma_4\psi \tag{40.}$$

and the scalar density

$$\rho_s = \bar{\psi}\psi. \tag{41.}$$

As the kinetic energy of the nucleon increases, ρ_s becomes much smaller than ρ. (Of course, in calculations of the nucleon-nucleon scattering, this has been included for a long time.)

The interaction is similarly divided. The exchange of σ mesons leads to the scalar mean field $\bar{U}$, which leads to an interaction $U\rho_s$ that is attractive. The interaction with ω mesons is the fourth component of a relativistic vector and leads to a repulsive interaction $\bar{V}\rho$.

The relativistic interaction has been calculated in a mean field approximation by Serot & Walecka (21) and Celenza & Shakin (22), and this program has been further evaluated by Horowitz & Serot (23). A detailed Brueckner-Hartree-Fock (BHF) calculation with relativity has been done by ter Haar & Malfliet (24) and also by Machleidt & Brockman (25). These authors find results for $\bar{V}\rho$ rather similar to those of Horowitz & Serot (23), i.e. the detailed Brueckner-Hartree-Fock calculation agrees quite well with the mean field. The resulting EOS is again quite stiff, even stiffer than that of Friedman & Pandharipande (20).

One disturbing result is the very small value of the effective mass of the nucleon, m^*/m. Horowitz & Serot used $m^*/m = 0.5$ while ter Haar & Malfliet have $m^*/m = 0.24$. It is very unlikely that the effective nucleon mass would become this small.

Ainsworth et al (26) use a simpler and more phenomenological approach. They start from the fact that the standard calculations by nonrelativistic BHF find the minimum of energy at about $2\rho_{nm}$, twice the observed density of nuclear matter. This has always been a trouble with the Brueckner-Hartree-Fock calculations, but Ainsworth et al made a virtue out of it.

They first note that the energy per nucleon vs density with two-body forces can be well represented by a quadratic expression:

$$E_2 = E_{nm}^{(2)} + \frac{K^{(2)}}{18}\left(\frac{\rho - 2\rho_{nm}}{2\rho_{nm}}\right)^2. \tag{42.}$$

The density scale here is $2\rho_{nm}$, so they hope to use this simple approximation up to $\rho \approx 4\rho_{nm}$.

They then remark that obviously the nonrelativistic two-body force calculation has to be corrected, both for relativity and for three-body forces. Relativity gives a repulsive correction

$$E_{rel} = B\left(\frac{\rho}{\rho_{nm}}\right)^{8/3}. \qquad 43.$$

Various explicit calculations give

$$1.6 < B < 3.6 \text{ MeV}. \qquad 44.$$

The correction in Equation 43 is very strongly dependent on density. It therefore shifts the minimum energy to a smaller density. With a reasonable value of B of about 4 MeV, the minimum is shifted to approximately ρ_{nm}.

Three-body forces arise from various sources. They are not strongly dependent on density, and therefore do not shift the saturation density appreciably. The simplest type of three-body forces, involving the (3–3) isobar, gives an attractive contribution

$$E_3 \approx -2(\rho/\rho_{nm})^{1.4} \text{ MeV}. \qquad 45.$$

This contribution is needed to bring about sufficient binding energy at the equilibrium density. A detailed discussion of these three-body and other corrections is given by Jackson, Rho & Krotschek (27).

These authors, as well as Ainsworth et al, find that there are very big correction terms from higher order diagrams. The diagrams involving nucleon loops give a strong repulsive contribution, which could be as large as +100 MeV. On the other hand, diagrams involving meson loops give attraction of about equal magnitude. Jackson et al calculate the ratio of these two and find that with suitable choice of the scalar meson mass this ratio can be 1 ± 0.05. In other words, these corrections nearly cancel. Using these three components of the energy (Equations 42, 43, and 45), Ainsworth et al construct an EOS for densities up to $6\rho_{nm}$. This is presented in Figure 5. They choose $B = 4.4$ MeV and the coefficient in Equation 45 to be -3.9 MeV. The basic energy is the two-body interaction, E_2. The relativistic correction rises rapidly near ρ_{nm} but then saturates at about +20 MeV. The three-body forces also saturate, at about -3 MeV. The total energy does not become very large, i.e. the equation of state is rather soft.

On a more fundamental level, Ainsworth et al (28) start from the σ model, introduced by Lee & Wick (29). With no nuclear matter present the field energy is

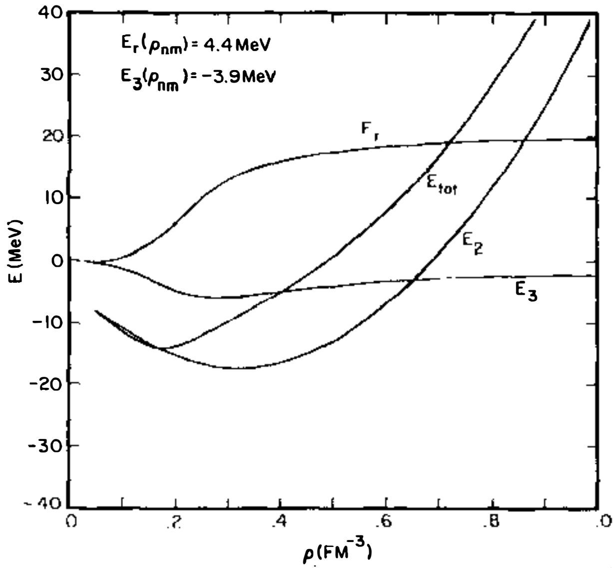

Figure 5 Contributions to energy of nuclear matter (in MeV per nucleon) vs density (nucleons per fm^3). E_2 = nonrelativistic two-body forces; E_r = relativistic corrections, E_3 = three-body forces, E_{tot} = sum of these. From Ainsworth et al (26).

$$V(\sigma) = \tfrac{1}{4}\lambda^2(\sigma^2 - f_\pi^2)^2. \tag{46.}$$

The minimum is at

$$\sigma_0 \simeq f_\pi = 93 \text{ MeV}, \tag{47.}$$

where the numerical value for f_π is only one possible choice. As nuclear matter is added, there is an energy of interaction,

$$\delta H = g\bar{\psi}\sigma\psi = g\rho_s\sigma. \tag{48.}$$

This corresponds to a free nucleon mass

$$M = g\sigma, \tag{49.}$$

where the coupling constant g has the dimension length squared. The nuclear matter shifts the minimum of the total energy to smaller values of σ, and we may write

$$\sigma \simeq f_\pi + \phi. \tag{50.}$$

The energy can be written as a polynomial in ϕ. At the same time as the minimum is shifted to negative ϕ, the curvature of the energy vs σ is also reduced. This means a reduction of the mass of the σ meson.

A reduction of m_σ^* means that the attractive nuclear force now has a longer range, and therefore becomes stronger. There is a much smaller effect on the mass of the repulsive ω meson so that the net attraction increases. This tends to make the nuclear EOS softer.

Ainsworth et al (28) argue that the reduction of m_σ in the nuclear medium is to be expected because ultimately nuclear matter should transform into a quark-gluon gas. This transition may be described by a parameter f_π^*, the value of f_π when matter is present. At the density where nuclear matter changes into a quark-gluon gas, f_π^* must go to zero. At this point the scalar particle becomes degenerate with the pion, so m_σ^* must become equal to m_π^*, the asterisks indicating the masses in the medium. Although m_π^* need not be quite as small as the vacuum value m_π, in detailed calculations (30) it is not much larger. Thus, m_σ^* must decrease substantially as f_π^*

Indeed, calculations in the Nambu-Jona-Lasinio formalism using the constituent quark model (30) are very suggestive. Since this model embodies the chiral invariance that is a property of the underlying Yang-Mills theory, and that unifies the isovector pions and the σ mesons into a four-vector in the (effective) nuclear-matter theory, results of these calculations should be much more general than the specific model used. In these calculations, the nucleon is made up of three constituent quarks. Hence $m_n = 3m_Q$ and, in the medium, $m_n^* = 3m_Q^*$, where m_Q is the constituent quark mass and m_Q^* the mass in the medium. It turns out that m_σ^* is close to $2m_Q^*$, since the interaction between the constituent quark and antiquark making up the σ is not very strong in the scalar channel. This is unlike the situation in the pseudoscalar channel, in which the interaction would bring the pion mass to zero at all nuclear matter densities were it not for the bare quark masses, which explicitly break chiral symmetry. Thus, it is found (30) that $m_\sigma^* \approx \frac{2}{3}m_n^*$, tracking the effective nucleon mass as it decreases with increasing density.

The σ model should describe the behavior of m_σ^* for some range of increasing density, but this model is not asymptotically free and need not give the transition to quark matter correctly. Calculations using a chiral bag model by Wüst, Brown & Jackson (31) show a rather smooth merging of nucleons to quarks as the pion cloud of the nucleon is squeezed out with increasing density.

The results of these considerations are that the repulsive nuclear loops saturate, but the attractive forces increase rapidly in magnitude because of the decrease of the σ mass. These behaviors follow the general rule that repulsive interactions tend to screen themselves so as to cut down the repulsion, while attractive interactions do not.

The authors use their scheme to calculate the effective interactions, $\bar{U}\rho_s$ and $\bar{V}\rho$. Remarkably, the results, especially for the attractive interaction, are nearly independent of the parameters used. Ainsworth et al (28) vary the coupling constant g_0 in the absence of nuclear matter from 10 to 13, and the mass of the free σ meson from 560 to 973 MeV. Regardless of these choices, $\bar{U}\rho_s$ is between 227 and 237 MeV, and $\bar{V}\rho$ between 91 and 114. The effective nucleon mass m^*/m is between 0.85 and 0.86, in agreement with other determinations. The properties of normal nuclear matter are also nearly independent of the assumptions, with ρ_{nm} between 0.144 and 0.185 fm^{-3}, the energy per particle between -10 and -23 MeV, and the compression modulus between 170 and 215 MeV, all close to accepted values.

The resulting EOS can be well represented in the form chosen by Baron, Cooperstein & Kahana (32):

$$E = \frac{K_0}{9\gamma(\gamma-1)}[u^{\gamma-1}+(\gamma-1)u^{-1}-\gamma], \tag{51.}$$

with $u = \rho/\rho_{nm}$ and $\gamma = 2.5$. This EOS is very suitable for the calculation of the collapse and reexpansion of supernovae, and makes it possible to have a successful prompt shock in that calculation.

7. MATTER RICH IN NEUTRONS

Müther, Prakash, and Ainsworth (33) have calculated the EOS of material rich in neutrons. They use Brueckner-Hartree-Fock theory, including relativity. Accordingly, they have a scalar and a vector interaction. They use a modern one-boson-exchange potential with a weak tensor force. It is weak because the tensor force due to exchange of pions is compensated, at short range, by a tensor force due to interaction with ρ meson. They use the version used by Ainsworth et al (26) (Section 6), of the interaction between nucleons, pointing out that a relativistic calculation with this interaction can account for the empirical saturation property of nuclear matter, particularly for the saturation density.

Using this approach, Müther et al calculate the energy of nuclear matter and that of neutron matter as a function of density, up to six times normal nuclear matter density. Of course, the energy of neutron matter does not have a minimum, and therefore has no place where the pressure is zero. They also calculate the symmetry energy at normal nuclear density and find 27 MeV, close to the empirical value 30 ± 2 MeV. They show that the excess of energy for unsymmetric nuclear matter is very nearly proportional to α^2, where

$$\alpha = 1-2Z/A. \tag{52.}$$

Müther et al obtain, in the course of their calculation, the effective mass of a nucleon, which decreases sharply with increasing density. However, they do not take into account that the masses of the important mesons, particularly σ and ω, also change with density. Therefore we do not consider their results for nuclear matter to be the best available, but prefer to use the results of Ainsworth et al (26). We use the Müther et al calculation for the *difference* of energy between neutron matter and nuclear matter.

Using the work of Müther et al, Ainsworth (private communication) has derived an interpolation formula for this difference, as follows:

$$\Delta E(\rho, \alpha) \equiv E(\rho, \alpha) - E_1(\rho) - \left[A + \frac{B}{1+Cu}\right]u\alpha^2, \qquad 53.$$

with

$$u = \rho/\rho_{\mathrm{nm}},\ A = 16\ \mathrm{MeV},\ B = 72\ \mathrm{MeV},\ C = 4,$$

and

$$E_1 \equiv E(\rho, 0) = E_0 + \frac{K_0}{9}\frac{u^{\gamma} - 1 - \gamma(u-1)}{u\gamma(\gamma-1)} \qquad \text{for } u \geq 1$$

$$E_1 = E_0(3u - 3u^2 + u^3) + (K_0/18)u(1-u)^2 \qquad \text{for } u < 1. \qquad 54.$$

From this one can derive the difference in pressure,

$$\frac{\Delta P}{\rho} = u\frac{\mathrm{d}\Delta E}{\mathrm{d}u} = \left[A + \frac{B}{(1+Cu)^2}\right]u, \qquad 55.$$

and the difference in compression modulus,

$$\Delta K = \frac{\mathrm{d}\Delta P}{\mathrm{d}\rho} = 2u\left[A + \frac{B}{(1+Cu)^3}\right]. \qquad 56.$$

These expressions represent very well the calculations by Müther et al. If $\alpha < 1$, one should add to the EOS of nuclear matter just $\alpha^2\Delta E$, etc.

In Table 6 we give the values of important quantities for various values of u, both for symmetric nuclear matter and for the difference ΔE, etc. The corrections are substantial; e.g. the pressure in neutron matter at $u = 3$ is twice as much as for symmetric nuclear matter. For the compression modulus, however, the difference is rather small.

It may be that the difference calculated by Müther et al is too large. This is because their EOS for nuclear matter is considerably stiffer than the EOS discussed in Section 6, which we consider more reliable. It is possible that the excessive stiffness carries over to the difference ΔE, etc.

Table 6 Comparison of P/ρ and K for neutron matter vs symmetric nuclear matter and the difference energy ΔE (Equation 53) as a function of density, according to Müther et al[a]

u		P/ρ		K	
$(=\rho/\rho_{nm})$	ΔE	nm	neut	nm	neut
1	30	0	19	220	253
2	48	23	57	625	690
3	65	48	97	1145	1240
4	81	76	141	1760	1890
5	97	108	189	2470	2630

[a] All quantities except u given in MeV; nm = nuclear matter, numbers are according to Section 6; neut = neutron matter.

However, at least at density near normal, the main effect is due to the tensor force, which gives a strong attraction between neutron and proton that is absent in the force between two neutrons. This effect becomes unimportant at higher density. We must await further calculations, similar to those described in Section 6, before we have a reliable EOS for neutron matter. However, for $\alpha = 1/3$, which is the situation in collapsing supernovae, the effects discussed in this section are small, and the approximation of Ainsworth et al should be sufficient.

We can calculate the equilibrium density of matter of $\alpha = 1/3$; it is

$$\rho(1/3)/\rho_{nm} = 0.915. \tag{57}$$

At this density, the pressure vanishes. The compression modulus at this density is little changed; it is $K = 196$ MeV, as compared with the assumed K for equilibrium nuclear matter of $\alpha = 0$, which is 220 MeV.

Using the formulae given above, Ainsworth has calculated the properties of neutron stars in equilibrium. He finds the results given in Table 7. Only

Table 7 Maximum gravitational masses and corresponding central densities, of neutron stars for various equations of state

γ	K_0 (MeV)	$M/M_\odot$[b]	ρ_c (fm^{-3})
MPA[a]		2.40	0.9
3	200	1.82	1.3
2.5	240	1.55	1.6
2.5	200	1.47	1.7

[a] According to the original paper by Müther et al (33).
[b] $M_\odot$ = mass of the sun.

the lines $\gamma = 3$ and 2.5 are calculated according to the prescription in this section. The γ and K_0 refer to the quantities in nuclear matter, Equation 51. A softer EOS, i.e. smaller γ or smaller K_0, gives a smaller maximum star mass and larger central density. The best observed neutron stars have masses between 1.4 and $1.5M_\odot$; so $\gamma = 2.5$ gives a satisfactory upper limit for the mass. However, a smaller value of γ would clearly be in contradiction with the observation.

8. TEMPERATURE DEPENDENCE

J. Lattimer has provided me with the temperature dependence of the EOS at high density. He assumes that the thermal correction in the nucleon-nucleon force is not important, for which there is ample evidence. Then the temperature correction to the energy enters only through the kinetic energy term. He writes

$$E_{\text{th}} \equiv E(T)-E(0) = n^{-1}\sum_i \frac{\hbar^2}{2m_i^*}[\tau_i(T)-\tau_i(0)], \tag{58.}$$

where $i = 1,2$ labels neutrons and protons. The number density of each species is given by

$$n_i = \frac{1}{2\pi^2}\left(\frac{2m_i^* T}{\hbar^2}\right)^{3/2} F_{1/2}(\mu_i'/T) \tag{59.}$$

where

$$F_k(x) = \int_0^\infty \frac{u^k \, \mathrm{d}u}{e^{u-x}+1}. \tag{60.}$$

The term τ_i is the kinetic energy density,

$$\tau_i = \frac{1}{2\pi^2}\left(\frac{2m_i^* T}{\hbar^2}\right)^{5/2} F_{3/2}(\mu_i'/T) \tag{61.}$$

and we can write

$$\mu_i' = \mu_i - V_i. \tag{62.}$$

Here μ_i is the chemical potential; this is corrected by the potential energy V_i of individual nucleons. This V_i is given by

$$V_i(T) - V_i(0) = \frac{\hbar^2}{2m_i^* n}[\tau_i(T)-\tau_i(0)]\left[-\frac{\mathrm{d}\ln m_i^*}{\mathrm{d}\ln n}\right] \tag{63.}$$

where m_i^* is the effective mass, as in Sections 4 and 6.

From the thermal energy, we can deduce the pressure

$$P_{\text{th}} = n \sum_i E_{\text{th},i} \left[\frac{2}{3} - \frac{\text{d} \ln m_i^*}{\text{d} \ln n} \right] \tag{64.}$$

and the entropy

$$S = T^{-1} \left[\sum_i \frac{5}{3} E_{\text{th},i} - \frac{n_i}{n} [\mu_i'(T) - \mu_i(0)] \right]. \tag{65.}$$

Above normal nuclear density, we are in the low temperature regime, $T < \mu_i$. In this case we can expand

$$\mu_i'(T) = \mu_i(0) - \frac{\pi^2}{12} \frac{T^2}{\mu_i(0)} \left[1 + \frac{3\pi^2}{20} \frac{T^2}{\mu_i^2(0)} \right] \tag{66.}$$

and

$$\mu_i(0) = \frac{\hbar^2}{2m_i^*} (3\pi^2 n_i)^{2/3} \tag{67.}$$

$$\tau_i(T) = \tau_i(0) + \frac{\pi^2}{4} \frac{T^2}{\mu_i(0)} n_i \frac{2m_i^*}{\hbar^2} \left[1 - \frac{3\pi^2}{20} \frac{T^2}{\mu_i^2(0)} \right]. \tag{68.}$$

Then the thermal energy becomes

$$E_{\text{th},i} = \frac{\pi^2}{4} \frac{n_i}{n} \frac{T^2}{\mu_i(0)} \left[1 - \frac{3\pi^2}{20} \frac{T^2}{\mu_i^2(0)} \right] \tag{69.}$$

and the entropy

$$S = \frac{\pi^2}{2} \frac{T}{n} \sum_i \frac{n_i}{\mu_i(0)} \left[1 - \frac{\pi^2}{10} \frac{T^2}{\mu_i^2(0)} \right]. \tag{70.}$$

It has been shown by explicit calculations, such as those reported in Section 6, that a good approximation is

$$\frac{m_i^*}{m} \approx \frac{1}{1 + \beta n} \tag{71.}$$

and

$$\frac{\text{d} \ln m^*}{\text{d} \ln n} = -\beta \frac{m^*}{m}. \tag{72.}$$

In this manner, one obtains the temperature corrections to the more important thermodynamic quantities.

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